

High Speed Communication Circuits and Systems
Lecture 2
Transmission Lines

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Maxwell's Equations

- **General form:**

$$\nabla \times E = -\mu \frac{dH}{dt} \quad (1)$$

$$\nabla \times H = J + \epsilon \frac{dE}{dt} \quad (2)$$

$$\nabla \cdot \epsilon E = \rho \quad (3)$$

$$\nabla \cdot \mu H = 0 \quad (4)$$

- **Assumptions for free space and transmission line propagation**

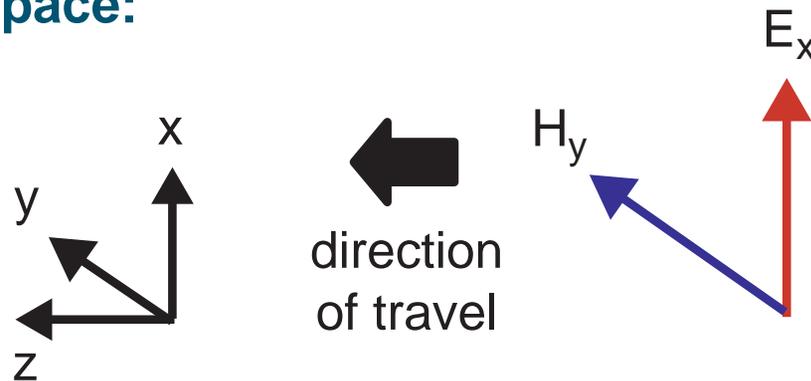
- No charge buildup: $\rho = 0$
- No free current: $J = 0$

- **Note: we'll only need Equations 1 and 2**

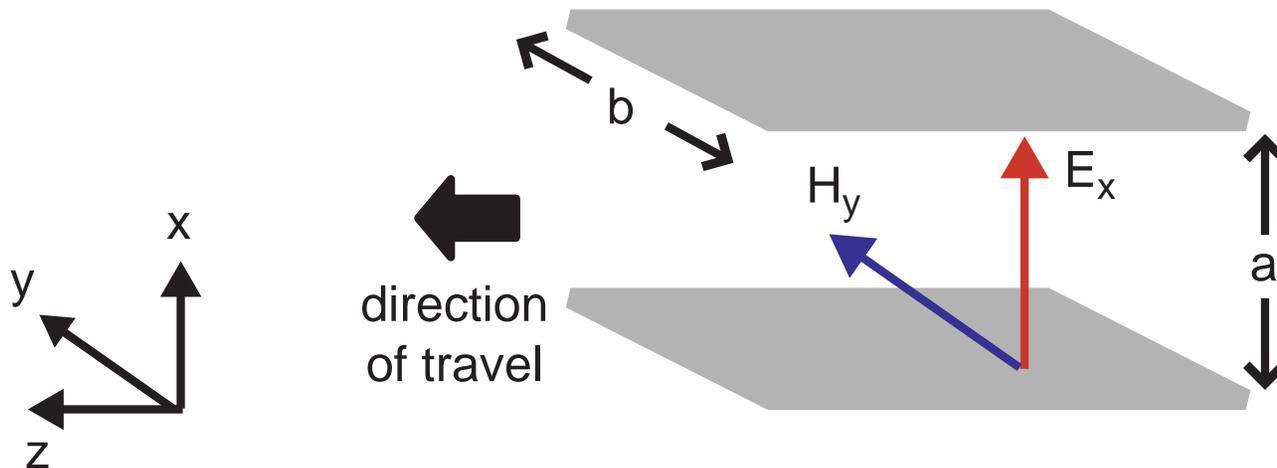
Assumptions

- **Orientation and direction**

- E field is in x-direction and traveling in z-direction
- H field is in y-direction and traveling in z-direction
- In freespace:



- **For transmission line (TEM mode)**



Solution

- Fields change only in time and in z-direction
 - Assume complex exponential solution

$$E = \hat{x}E_x(z, t) = \hat{x}E_0e^{-jkz}e^{j\omega t}$$

$$H = \hat{y}H_y(z, t) = \hat{y}H_0e^{-jkz}e^{j\omega t}$$

Solution

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$$H = \hat{y}H_y(z, t) = \hat{y}H_0e^{-jkz}e^{j\omega t}$$

- Implications:

$$\frac{dE_x(z, t)}{dz} = -jkE_x(z, t), \quad \frac{dE_x(z, t)}{dt} = j\omega E_x(z, t)$$

$$\frac{dH_y(z, t)}{dz} = -jkH_y(z, t), \quad \frac{dH_y(z, t)}{dt} = j\omega H_y(z, t)$$

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- Implications:

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$$\frac{dH_y(z, t)}{dz} = -jkH_y(z, t), \quad \frac{dH_y(z, t)}{dt} = j\omega H_y(z, t)$$

But, what is the value of k ?

Evaluate Curl Operations in Maxwell's Formula

- **Definition**

$$\nabla \times E = \hat{x} \left(\frac{dE_z}{dy} - \frac{dE_y}{dz} \right) + \hat{y} \left(\frac{dE_x}{dz} - \frac{dE_z}{dx} \right) + \hat{z} \left(\frac{dE_y}{dx} - \frac{dE_x}{dy} \right)$$

$$\nabla \times H = \hat{x} \left(\frac{dH_z}{dy} - \frac{dH_y}{dz} \right) + \hat{y} \left(\frac{dH_x}{dz} - \frac{dH_z}{dx} \right) + \hat{z} \left(\frac{dH_y}{dx} - \frac{dH_x}{dy} \right)$$

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$$\nabla \times H = \hat{x} \left(\frac{dH_z}{dy} - \frac{dH_y}{dz} \right) + \hat{y} \left(\frac{dH_x}{dz} - \frac{dH_z}{dx} \right) + \hat{z} \left(\frac{dH_y}{dx} - \frac{dH_x}{dy} \right)$$

- **Given the previous assumptions**

$$\nabla \times E = \hat{y} \frac{dE_x(z, t)}{dz} = -\hat{y} jk E_x(z, t)$$

$$\nabla \times H = -\hat{x} \frac{dH_y(z, t)}{dz} = \hat{x} jk H_y(z, t)$$

Now Put All the Pieces Together

- **Solve Maxwell's Equation (1)**

$$\begin{aligned}\nabla \times E &= -\mu \frac{dH}{dt} \Rightarrow -\hat{y} jk E_x(z, t) = -\hat{y} \mu j\omega H_y(z, t) \\ &\Rightarrow \frac{E_x(z, t)}{H_y(z, t)} = \frac{\mu\omega}{k} \quad (\text{intrinsic impedance})\end{aligned}$$

Now Put All the Pieces Together

- Solve Maxwell's Equations (1) and (2)

$$\begin{aligned}\nabla \times E &= -\mu \frac{dH}{dt} \Rightarrow -\hat{y} jk E_x(z, t) = -\hat{y} \mu j\omega H_y(z, t) \\ &\Rightarrow \frac{E_x(z, t)}{H_y(z, t)} = \frac{\mu\omega}{k} \quad (\text{intrinsic impedance})\end{aligned}$$

$$\begin{aligned}\nabla \times H &= \epsilon \frac{dE}{dt} \Rightarrow \hat{x} jk H_y(z, t) = \hat{x} \epsilon j\omega E_x(z, t) \\ &\Rightarrow H_y(z, t) = \frac{\epsilon\omega}{k} E_x(z, t) = \frac{\epsilon\omega}{k} \left(\frac{\mu\omega}{k} \right) H_y(z, t) \\ &\Rightarrow \frac{\epsilon\omega}{k} \left(\frac{\mu\omega}{k} \right) = 1 \Rightarrow \boxed{k = \omega \sqrt{\mu\epsilon}}\end{aligned}$$


Now Put All the Pieces Together

- Solve Maxwell's Equations (1) and (2)

$$\begin{aligned}\nabla \times E &= -\mu \frac{dH}{dt} \Rightarrow -\hat{y} jk E_x(z, t) = -\hat{y} \mu j\omega H_y(z, t) \\ &\Rightarrow \frac{E_x(z, t)}{H_y(z, t)} = \frac{\mu\omega}{k} \quad (\text{intrinsic impedance})\end{aligned}$$

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$$\Rightarrow \text{intrinsic impedance} = \frac{\mu\omega}{k} = \frac{\mu\omega}{\omega \sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}$$

Connecting to the Real World

- **Current solution is complex**

$$E = \hat{x} E_x(z, t) = \hat{x} E_o e^{-jkz} e^{j\omega t} = \hat{x} E_o e^{-j(\omega t - kz)}$$

- **But the following complex solution is also valid**

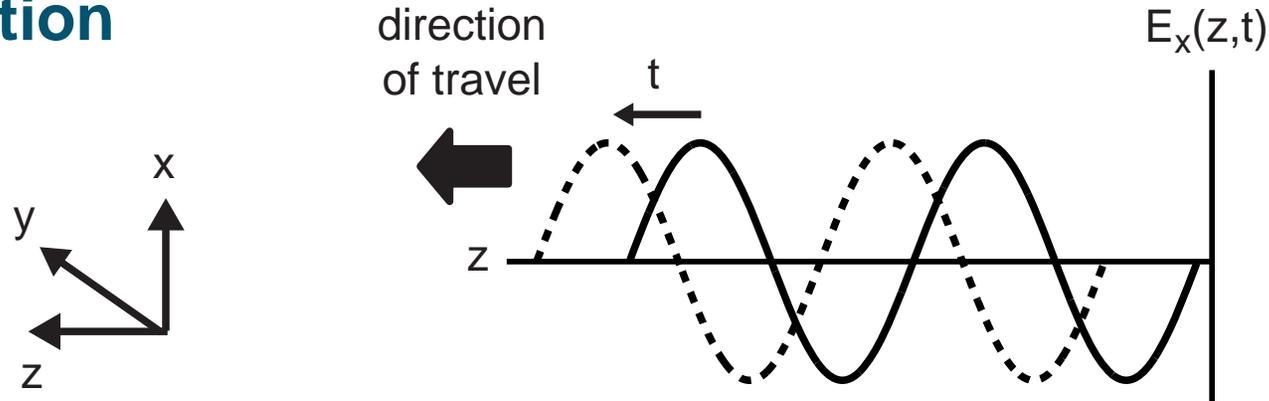
$$E = \hat{x} E_x(z, t) = \hat{x} E_o e^{j(\omega t - kz)}$$

- **And adding them together is also a valid solution that is now real-valued**

$$\begin{aligned} E &= \hat{x} E_o (e^{j(\omega t - kz)} + e^{-j(\omega t - kz)}) \\ &= \hat{x} 2E_o \cos(\omega t - kz) \end{aligned}$$

Calculating Propagation Speed

- The resulting cosine wave is a function of time AND position



$$E_x(z, t) = \hat{x} 2E_o \cos(\omega t - kz)$$

- Consider “riding” one part of the wave

$$-kz + \omega t = \text{constant (choose 0)} \Rightarrow z = \frac{\omega t}{k}$$

- Velocity calculation

$$\frac{dz}{dt} = \frac{d}{dt} \left(\frac{\omega t}{k} \right) = \frac{\omega}{k} = \frac{\omega}{\omega \sqrt{\mu \epsilon}} = \boxed{\frac{1}{\sqrt{\mu \epsilon}}}$$

FreeSpace Values

- **Constants**

$$\epsilon = \epsilon_0 = \frac{1}{36\pi} \times 10^{-9} \text{ F/m}$$

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

- **Impedance**

$$\sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \text{ Ohms}$$

- **Propagation speed**

$$\frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 30 \times 10^9 \text{ cm/s}$$

- **Wavelength of 30 GHz signal**

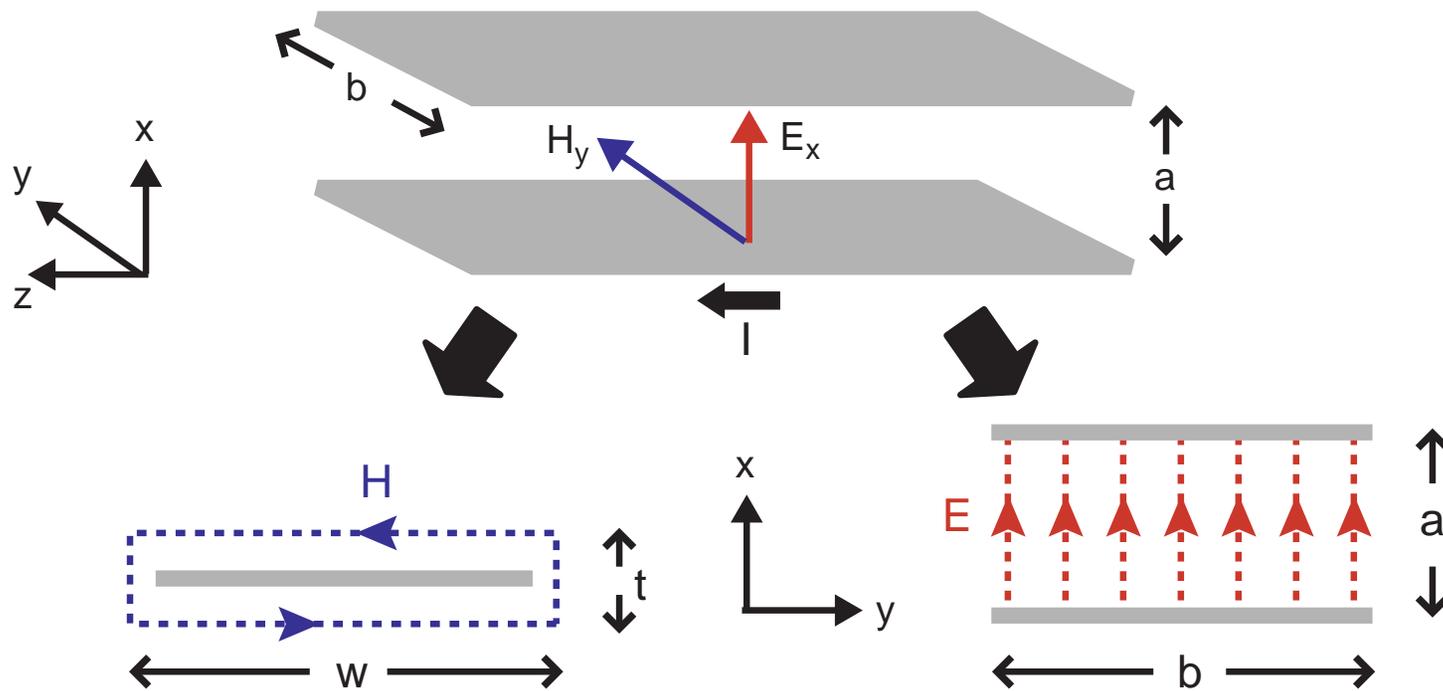
$$\lambda = \frac{T}{\sqrt{\mu\epsilon}} = \frac{1}{f\sqrt{\mu_0\epsilon_0}} = 1 \text{ cm}$$

Voltage and Current

- Definitions:**

$$V = \int_{C_t} E \cdot dl \quad (\text{path integral})$$

$$I = \oint_{C_o} H \cdot dl \quad (\text{contour integral})$$

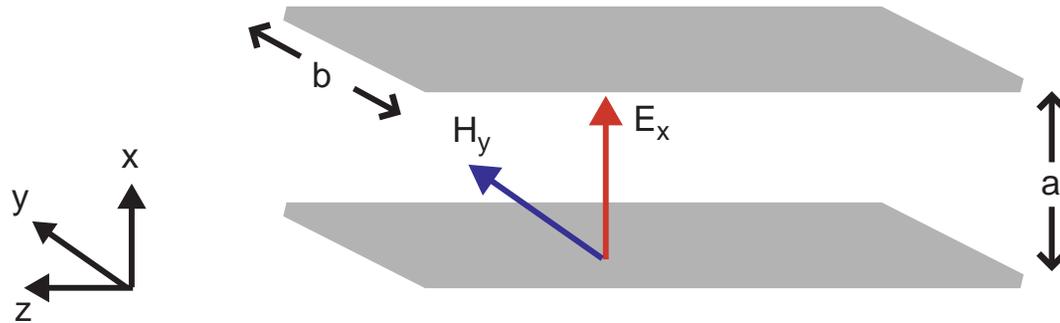


$$I = (2w + 2t)H$$

$$V = aE$$

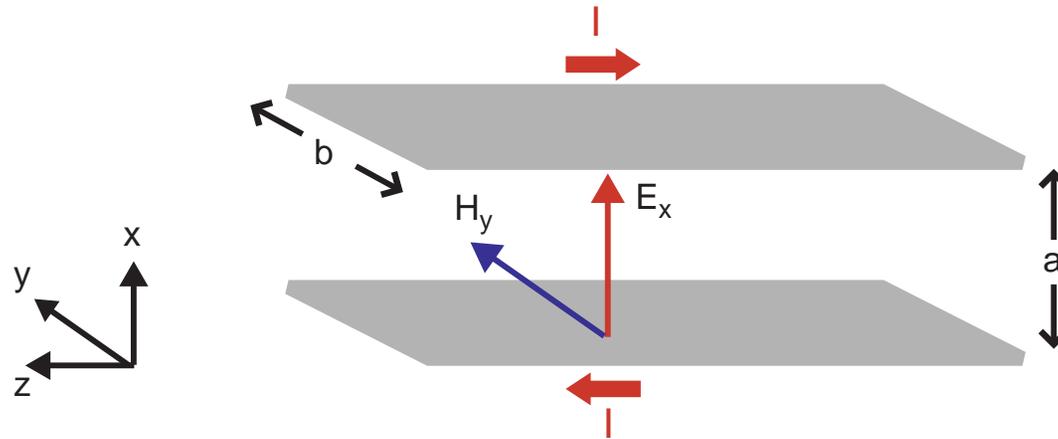
Parallel Plate Waveguide

- E-field and H-field are influenced by plates



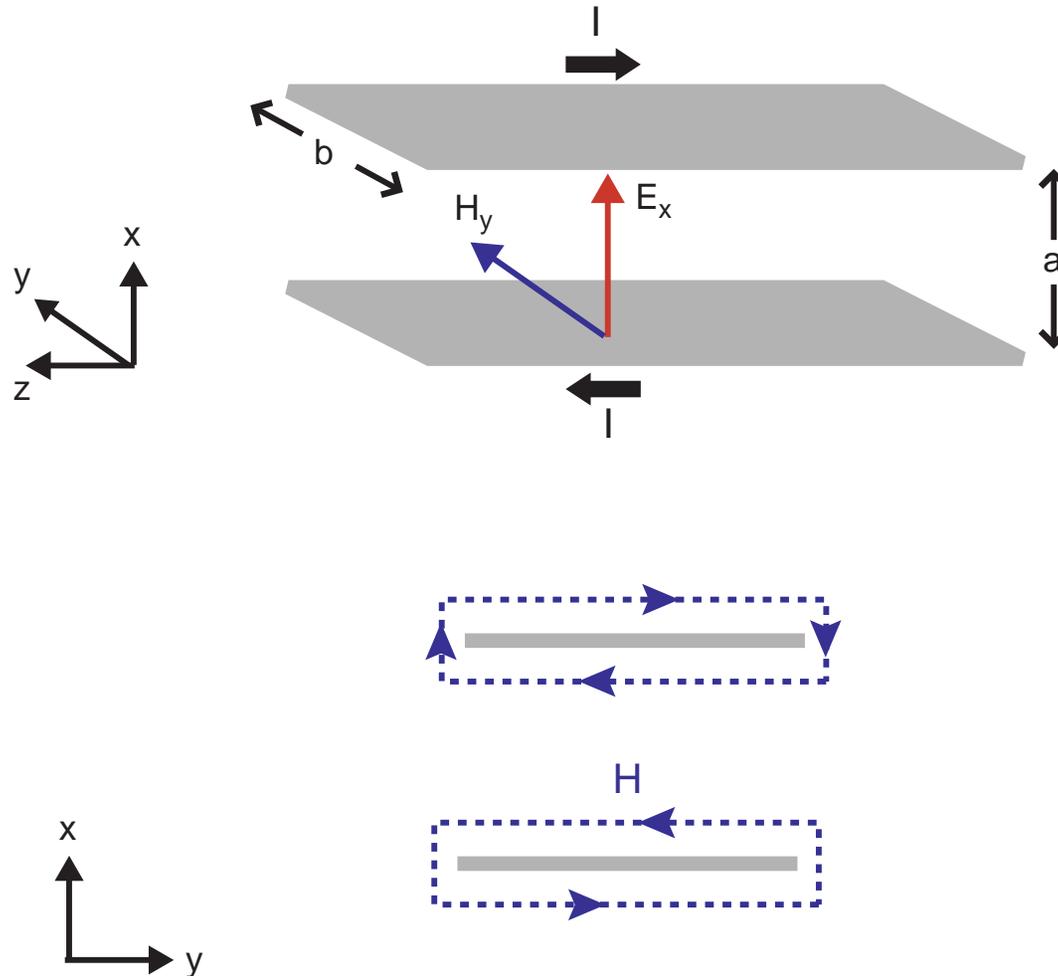
Current and H-Field

- Assume that (AC) current is flowing



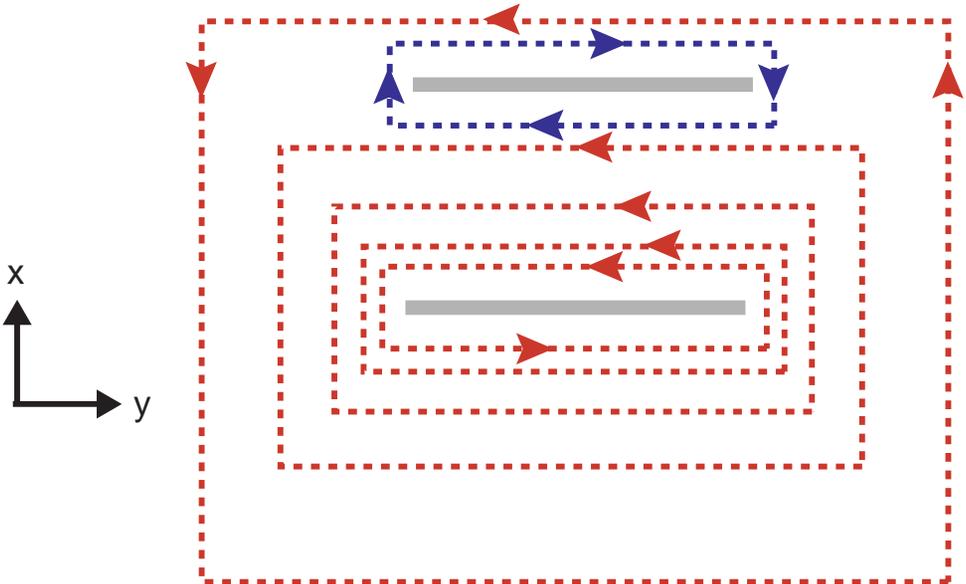
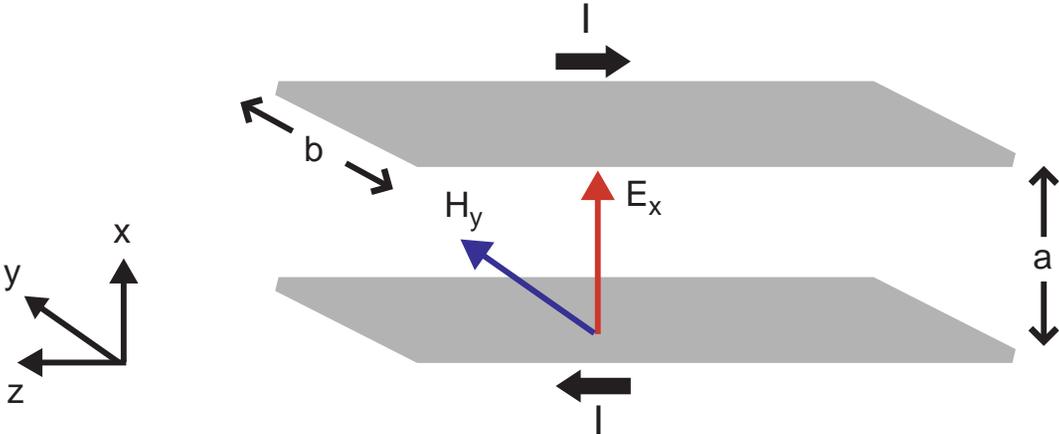
Current and H-Field

- Current flowing down waveguide influences H-field



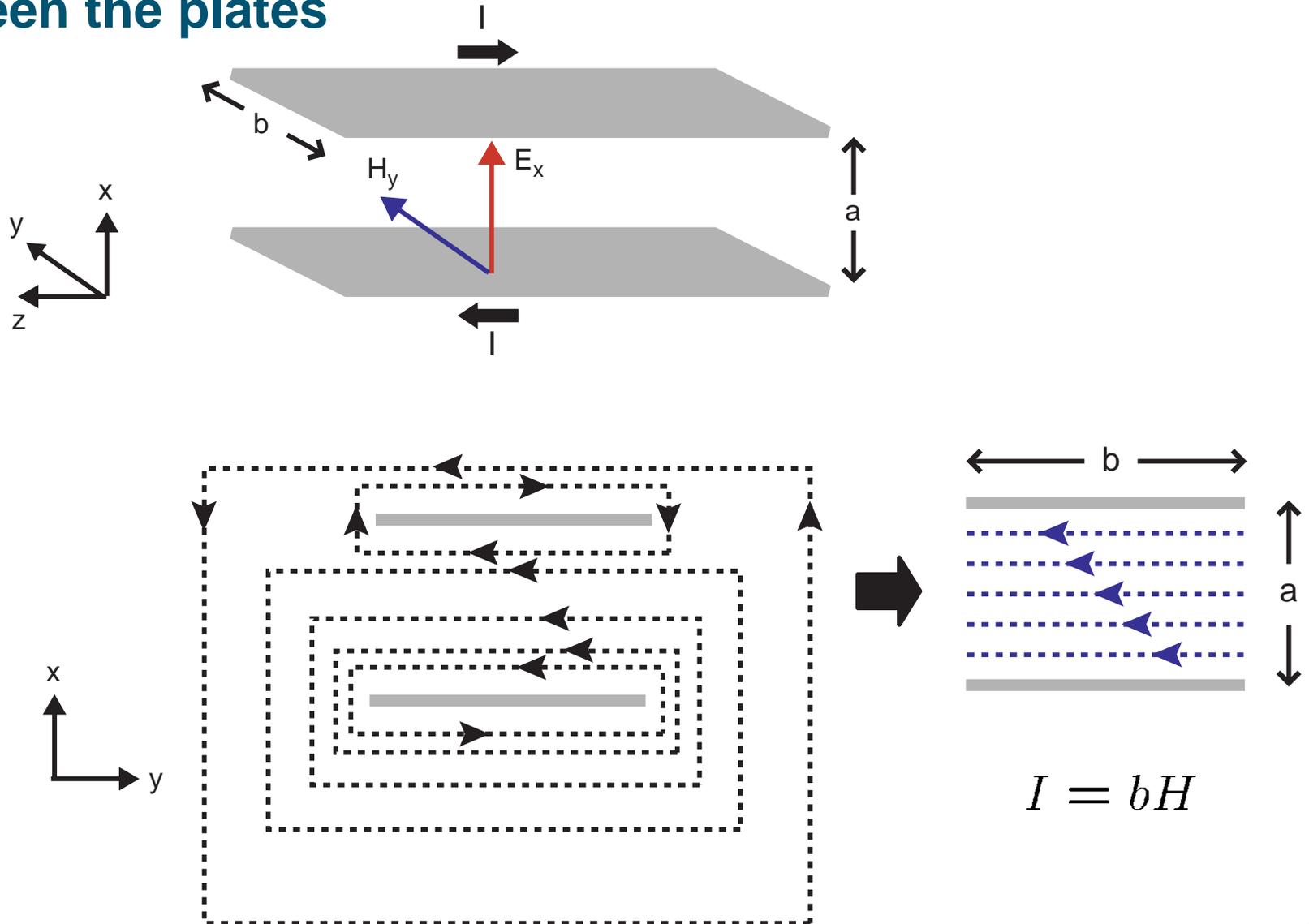
Current and H-Field

- Flux from one plate interacts with flux from the other plate



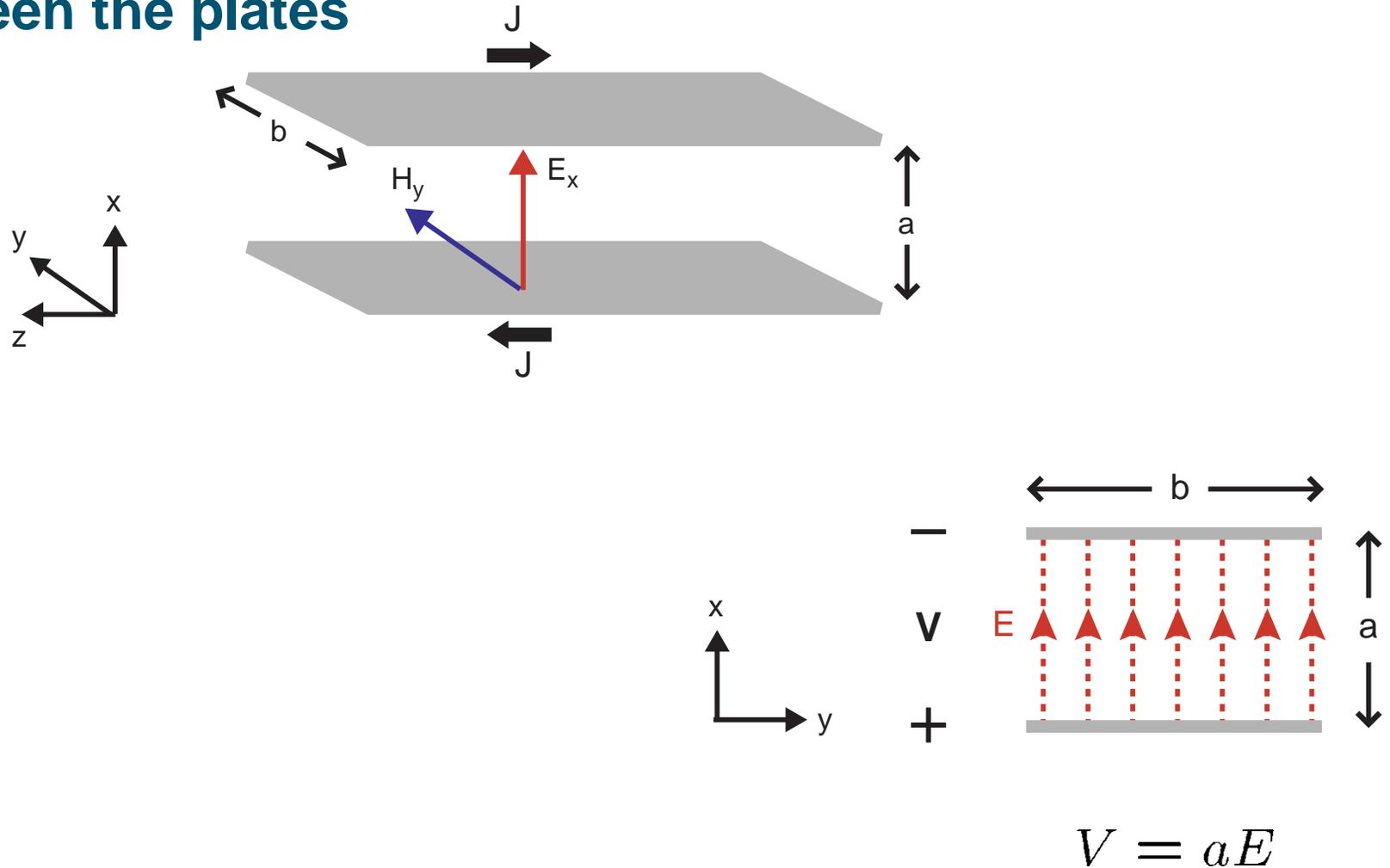
Current and H-Field

- Approximate H-Field to be uniform and restricted to lie between the plates



Voltage and E-Field

- Approximate E-field to be uniform and restricted to lie between the plates



Back to Maxwell's Equations

- From previous analysis

$$\nabla \times E = -\mu \frac{dH}{dt} \Rightarrow jkE_x(z, t) = j\omega\mu H_y(z, t)$$

$$\nabla \times H = \epsilon \frac{dE}{dt} \Rightarrow jkH_y(z, t) = j\omega\epsilon E_x(z, t)$$

- These can be equivalently written as

$$jk(aE_x(z, t)) = j\omega\mu \frac{a}{b} (bH_y(z, t)) \Rightarrow jkV(z, t) = j\omega LI(z, t)$$

$$jk(bH_y(z, t)) = j\omega\epsilon \frac{b}{a} (aE_x(z, t)) \Rightarrow jkI(z, t) = j\omega CV(z, t)$$

- Where

$$L = \mu \frac{a}{b} \text{ (inductance per unit length - H/m)}$$

$$C = \epsilon \frac{b}{a} \text{ (capacitance per unit length - F/m)}$$

Wave Equation for Transmission Line (TEM)

- Key formulas

$$jkV(z, t) = j\omega LI(z, t) \quad (1)$$

$$jkI(z, t) = j\omega CV(z, t) \quad (2)$$

- Substitute (2) into (1)

$$jkV(z, t) = j\omega L \left(\frac{\omega}{k} CV(z, t) \right) \Rightarrow (k^2 - \omega^2 LC)V(z, t) = 0$$

$$\Rightarrow k = \omega\sqrt{LC}$$

- Characteristic impedance (use Equation (1))

$$\frac{V(z, t)}{I(z, t)} = \frac{\omega L}{k} = \frac{\omega L}{\omega\sqrt{LC}} = \sqrt{\frac{L}{C}}$$

Connecting to the Real World

- **Current solution is complex**

$$V(z, t) = V_o e^{-jkz} e^{j\omega t} = V_o e^{-j(\omega t - kz)}$$

- **But the following solution is also valid**

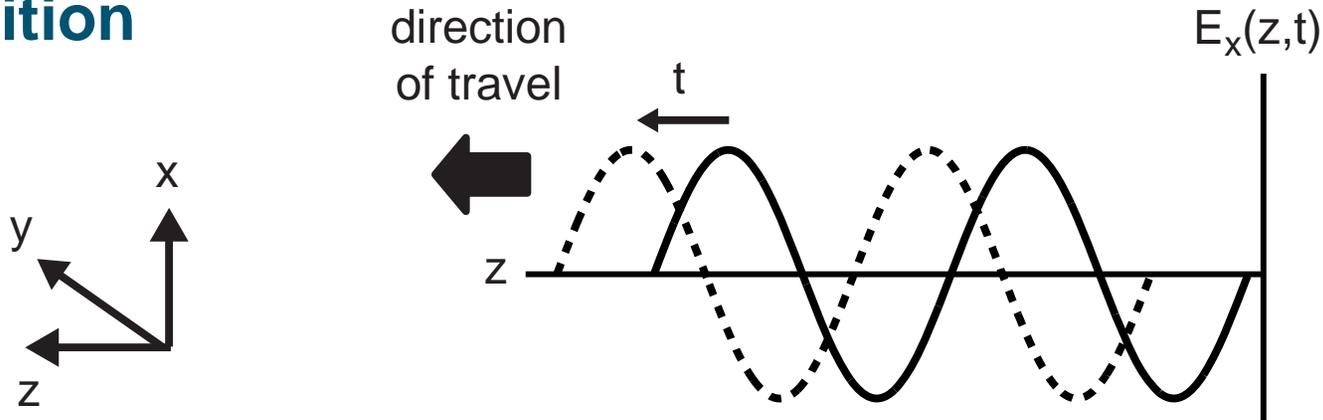
$$V(z, t) = V_o e^{j(\omega t - kz)}$$

- **And adding them together is also a valid solution**

$$\begin{aligned} V &= V_o (e^{j(\omega t - kz)} + e^{-j(\omega t - kz)}) \\ &= 2V_o \cos(\omega t - kz) \end{aligned}$$

Calculating Propagation Speed

- The resulting cosine wave is a function of time AND position



$$V(z, t) = 2V_o \cos(\omega t - kz)$$

- Consider “riding” one part of the wave

$$-kz + \omega t = \text{constant (choose 0)} \Rightarrow z = \frac{\omega t}{k}$$

- Velocity calculation

$$\frac{dz}{dt} = \frac{d}{dt} \left(\frac{\omega t}{k} \right) = \frac{\omega}{k} = \frac{\omega}{\omega \sqrt{LC}} = \boxed{\frac{1}{\sqrt{LC}}}$$

Integrated Circuit Values

- **Constants**

$$\epsilon = \epsilon_r \epsilon_0 \quad (\epsilon_r = 3.9, 11.7, 4.4 \text{ in } SiO_2, Si, FR4, \text{ respectively})$$

$$\mu = \mu_r \mu_0 \quad (\mu_r = 1 \text{ for the above materials})$$

- **Impedance (geometry dependant)**

$$\sqrt{\frac{L}{C}} = \sqrt{\frac{\mu(a/b)}{\epsilon(b/a)}} = \sqrt{\frac{\mu}{\epsilon}} \left(\frac{a}{b}\right)$$

- **Propagation speed (geometry independent)**

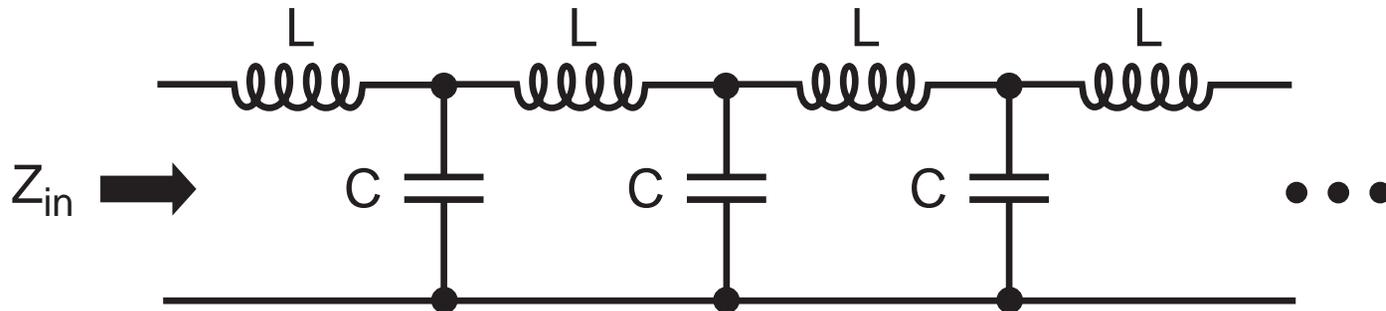
$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu(a/b)\epsilon(b/a)}} = \frac{1}{\sqrt{\mu\epsilon}} = 30 \times 10^9 \text{ cm/s}$$

- **Wavelength of 30 GHz signal in silicon dioxide**

$$\lambda = \frac{T}{\sqrt{\mu\epsilon}} = \frac{1}{f \sqrt{3.9\mu_0\epsilon_0}} = 1/2 \text{ cm}$$

LC Network Analogy of Transmission Line (TEM)

- LC network analogy



- Calculate input impedance

$$Z_{in} = sL + (1/sC) \parallel Z_{in} = sL + \frac{Z_{in}}{1 + Z_{in}sC}$$

$$\Rightarrow Z_{in}^2 - sLZ_{in} - L/C = 0$$

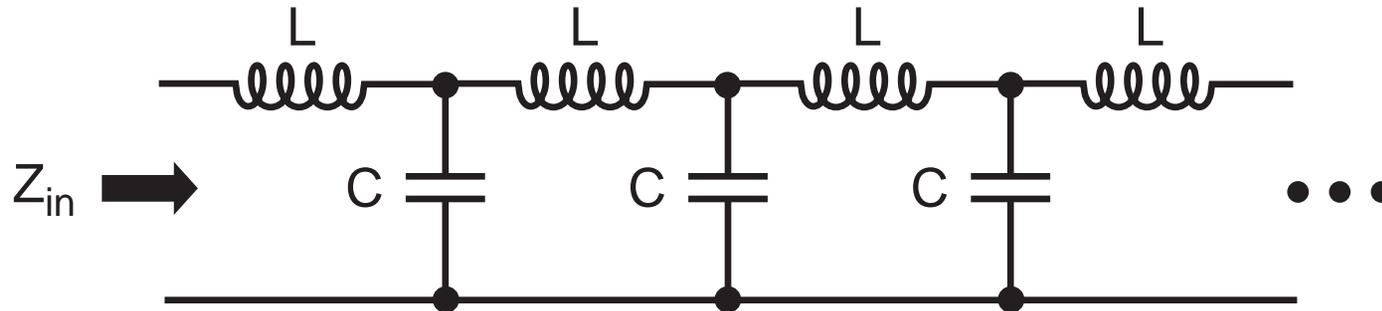
$$\Rightarrow Z_{in} = \frac{sL}{2} \left(1 \pm \sqrt{1 + \frac{4}{s^2LC}} \right)$$

$$\text{for } |s| \ll \frac{1}{LC} \Rightarrow Z_{in} \approx \frac{sL}{2} \left(1 \pm \frac{2}{s\sqrt{LC}} \right) \approx \sqrt{\frac{L}{C}}$$

How are Lumped LC and Transmission Lines Different?

- In transmission line, L and C values are infinitely small

- It is always true that $|s| \ll \frac{1}{LC}$



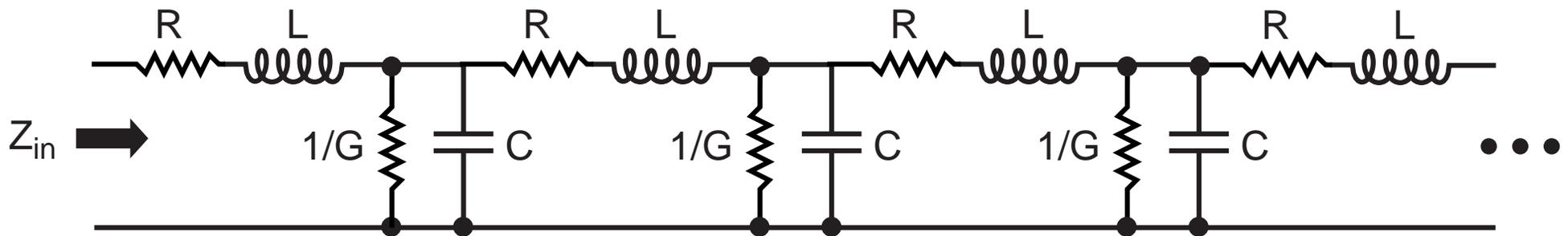
- For lumped LC, L and C have finite values

- Finite frequency range for $|s| \ll \frac{1}{LC}$

$$Z_{in} = \frac{sL}{2} \left(1 \pm \sqrt{1 + \frac{4}{s^2 LC}} \right) \Rightarrow \text{want } |s| < \frac{2}{\sqrt{LC}} \text{ for real } Z_{in}$$

Lossy Transmission Lines

- Practical transmission lines have losses in their conductor and dielectric material
 - We model such loss by including resistors in the LC model



- The presence of such losses has two effects on signals traveling through the line
 - Attenuation
 - Dispersion (i.e., bandwidth degradation)
- See Chapter 5 of Thomas Lee's book for analysis