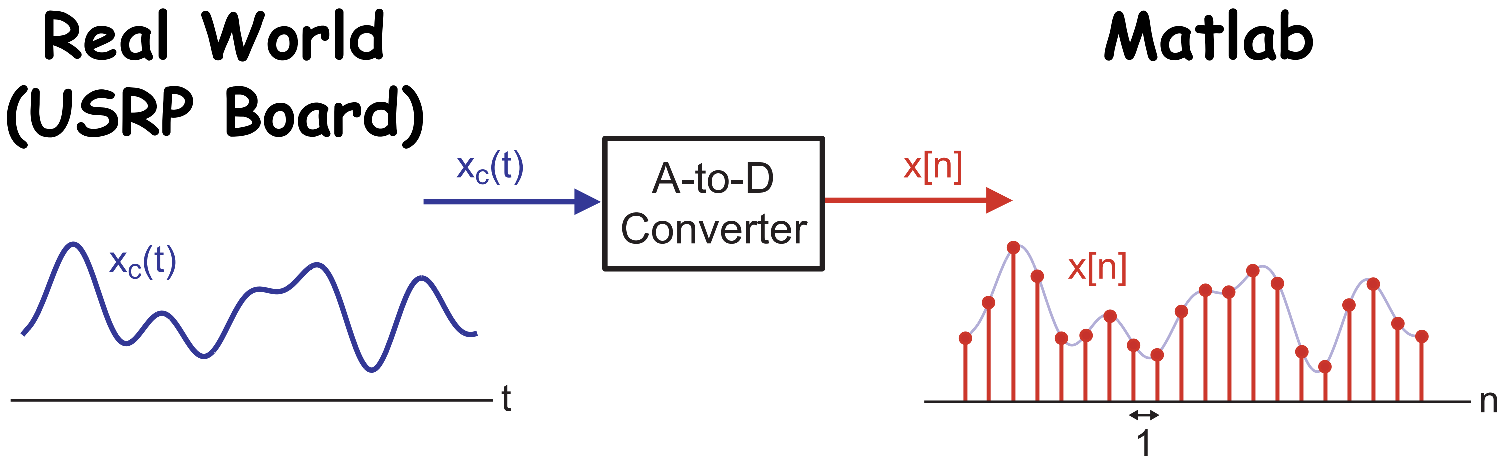
# Sampling Continuous-Time Signals

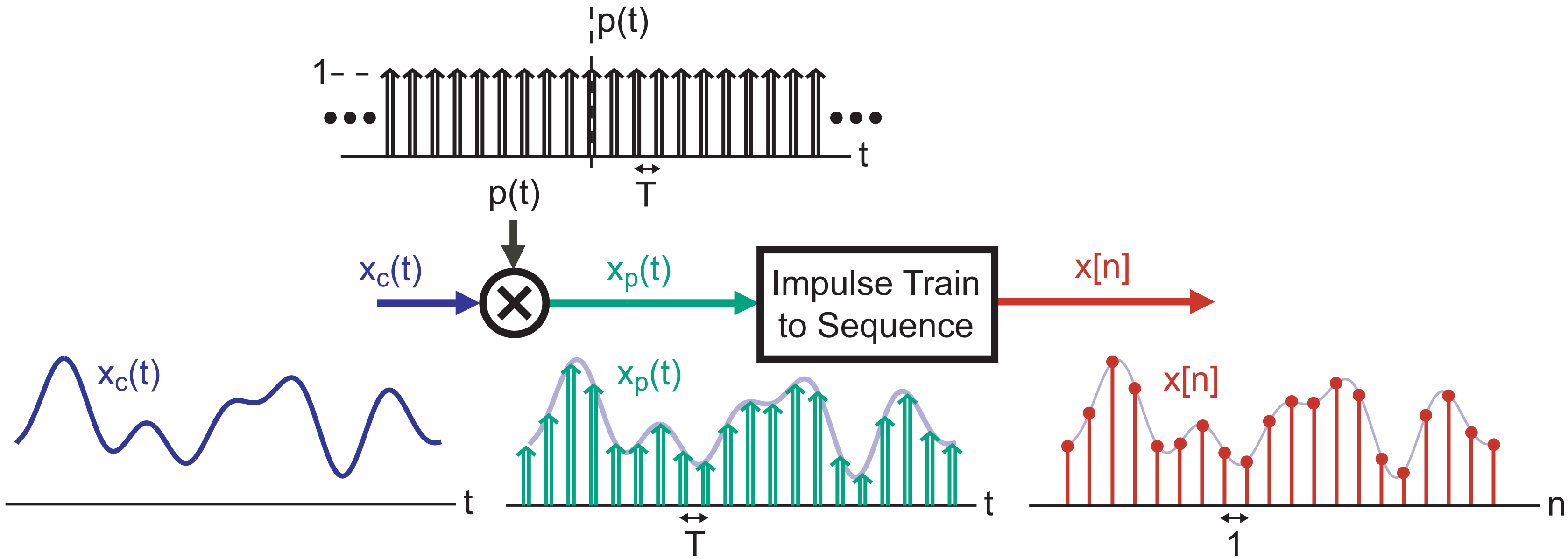
- **Impulse train and its Fourier Transform**
- **Impulse samples versus discrete-time sequences**
- **Aliasing and the Sampling Theorem**
- **Anti-alias filtering**
- **Comparison of FT, DTFT, Fourier Series**

# The Need for Sampling




- The boundary between *analog* and *digital*
  - Real world is filled with *continuous-time signals*
  - Computers (i.e. Matlab) operate on *sequences*
- Crossing the analog-to-digital boundary requires sampling of the continuous-time signals
- Key questions
  - How do we analyze the sampling process?
  - What can go wrong?

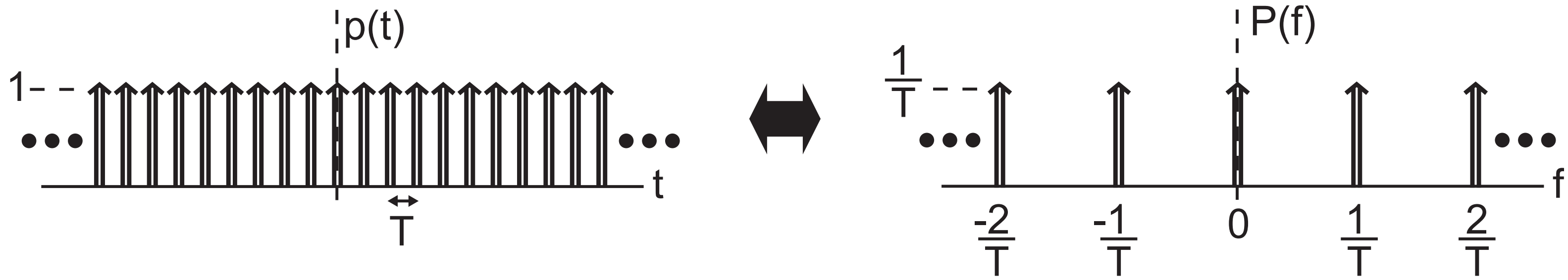
# An Analytical Model for Sampling




- **Two step process**
  - Sample continuous-time signal every *T* seconds
    - Model as *multiplication* of signal with *impulse train*
  - Create sequence from amplitude of scaled impulses
    - Model as *rescaling* of time axis (*T* goes to 1)
    - Notation: replace impulses with stem symbols

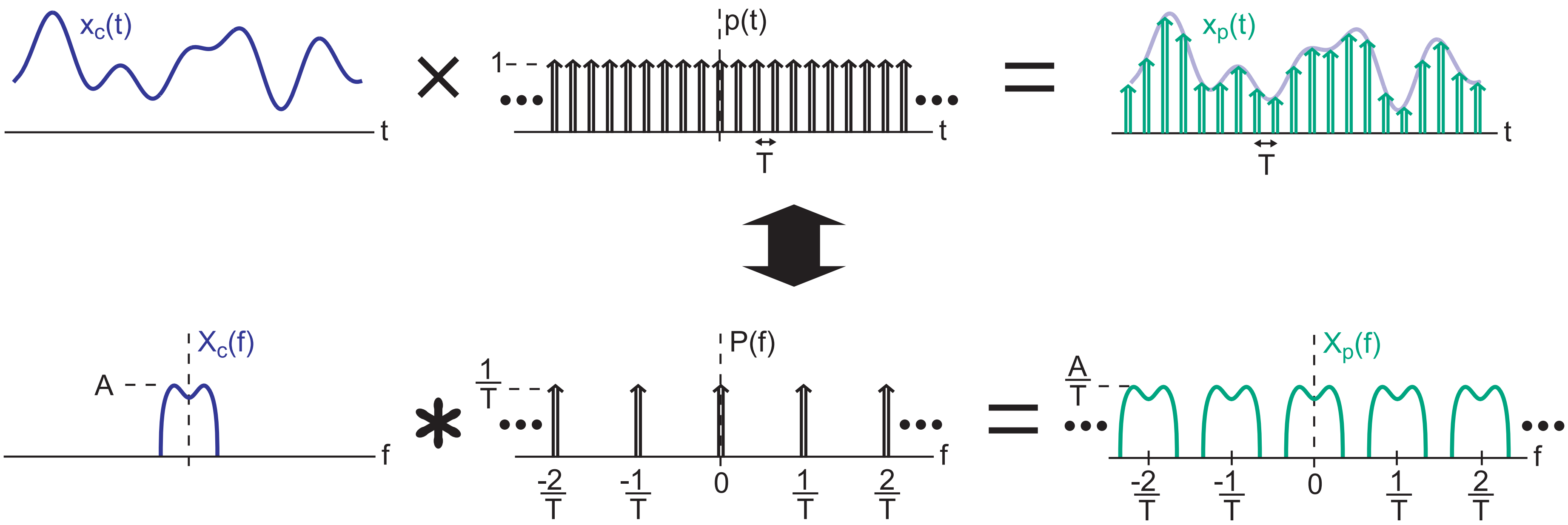
**Can we model this in the frequency domain?**

# Fourier Transform of Impulse Train




- **Impulse train in time corresponds to impulse train in frequency**
  - **Spacing in time of *T* seconds corresponds to spacing in frequency of *1/T* Hz**
  - **Scale factor of *1/T* for impulses in frequency domain**
  - **Note: this is painful to derive, so we won't …**
- **The above transform pair allows us to see the following with *pictures***
  - **Sampling operation in frequency domain**
  - **Intuitive comparison of FT, DTFT, and Fourier Series**

# Frequency Domain View of Sampling

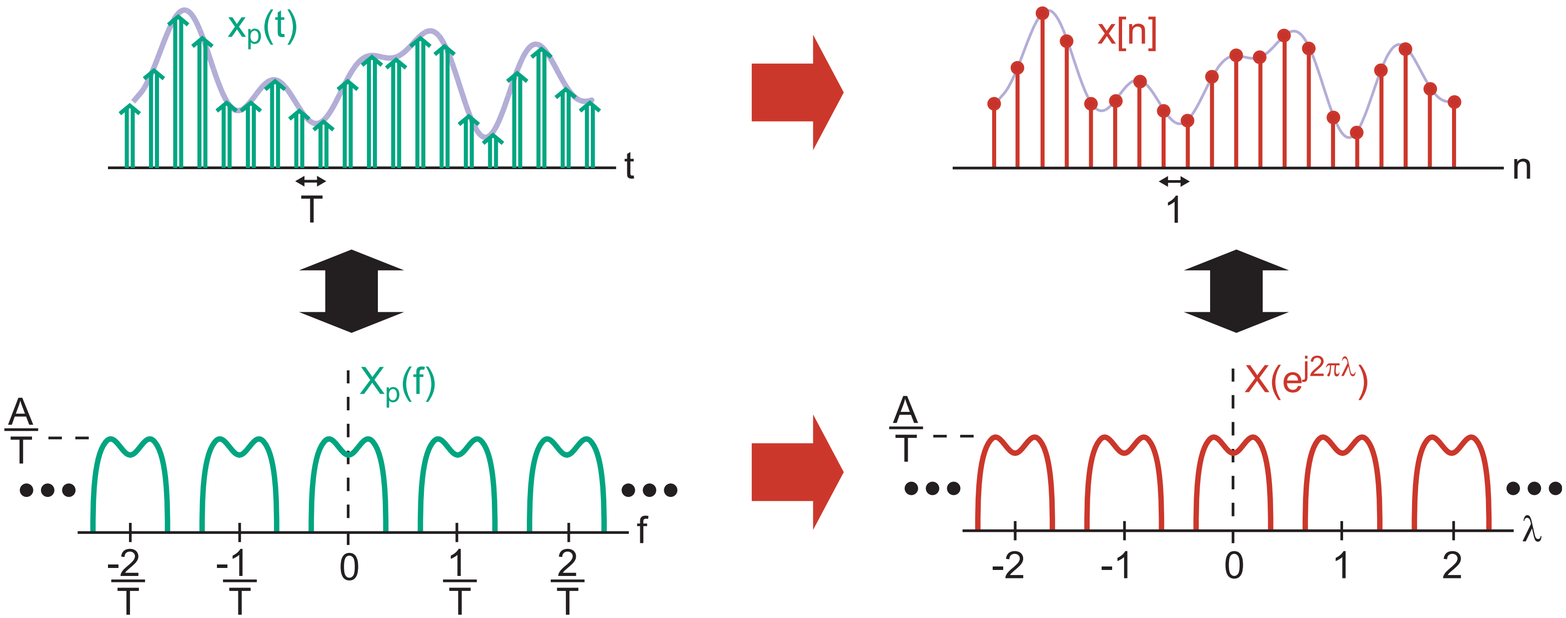



- Recall that *multiplication* in *time* corresponds to *convolution* in *frequency*

$$x(t)y(t) \Leftrightarrow X(f) * Y(f)$$

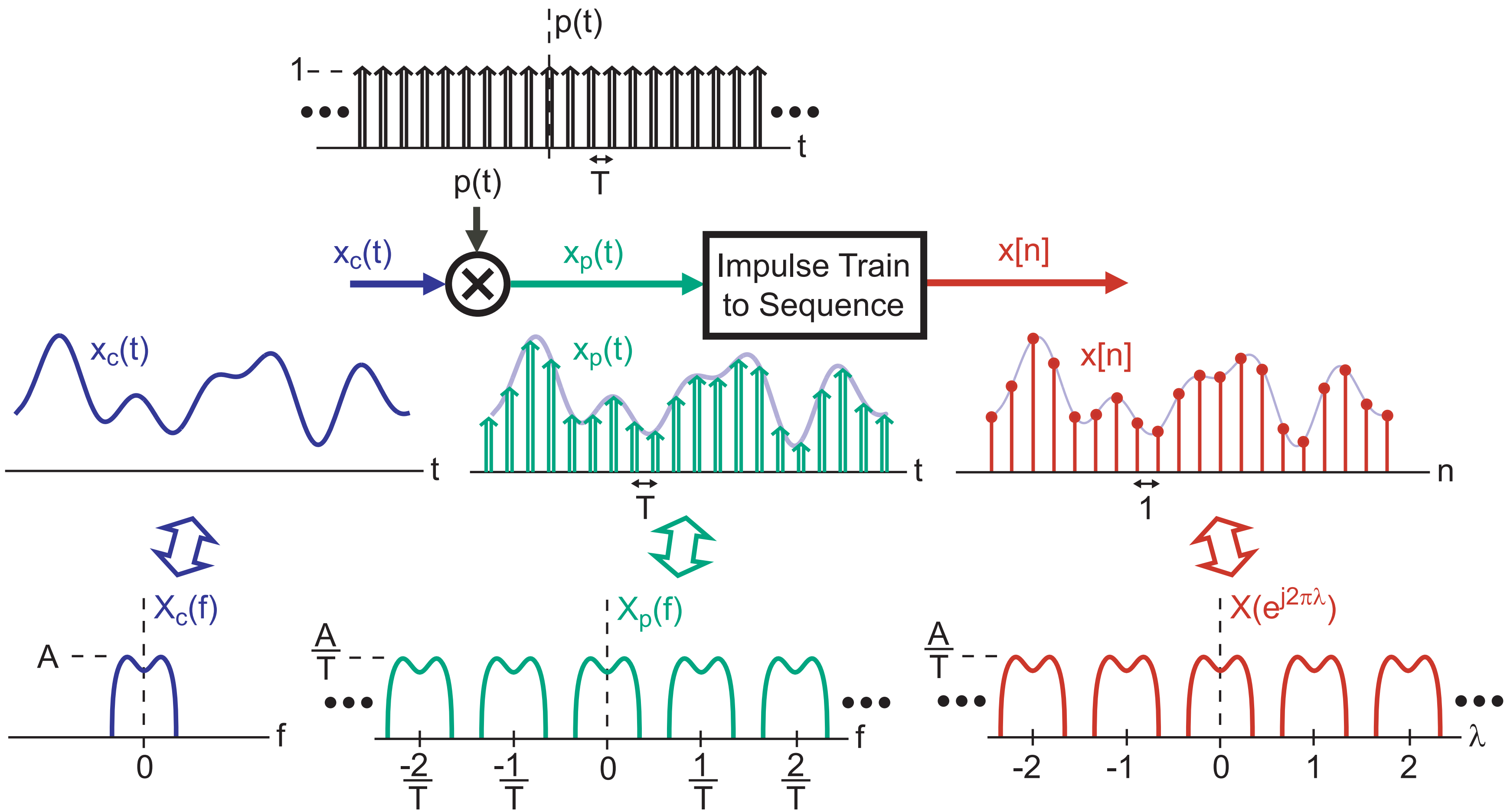
- We see that sampling in time leads to a *periodic* Fourier Transform with period *1/T*

# Frequency Domain View of Output Sequence




- **Scaling in time leads to scaling in frequency**
  - **Compression/expansion in time leads to expansion/ compression in frequency**
- **Conversion to sequence amounts to *T* going to 1**
  - **Resulting Fourier Transform is now periodic with period 1**
  - **Note that we are now essentially dealing with the DTFT**

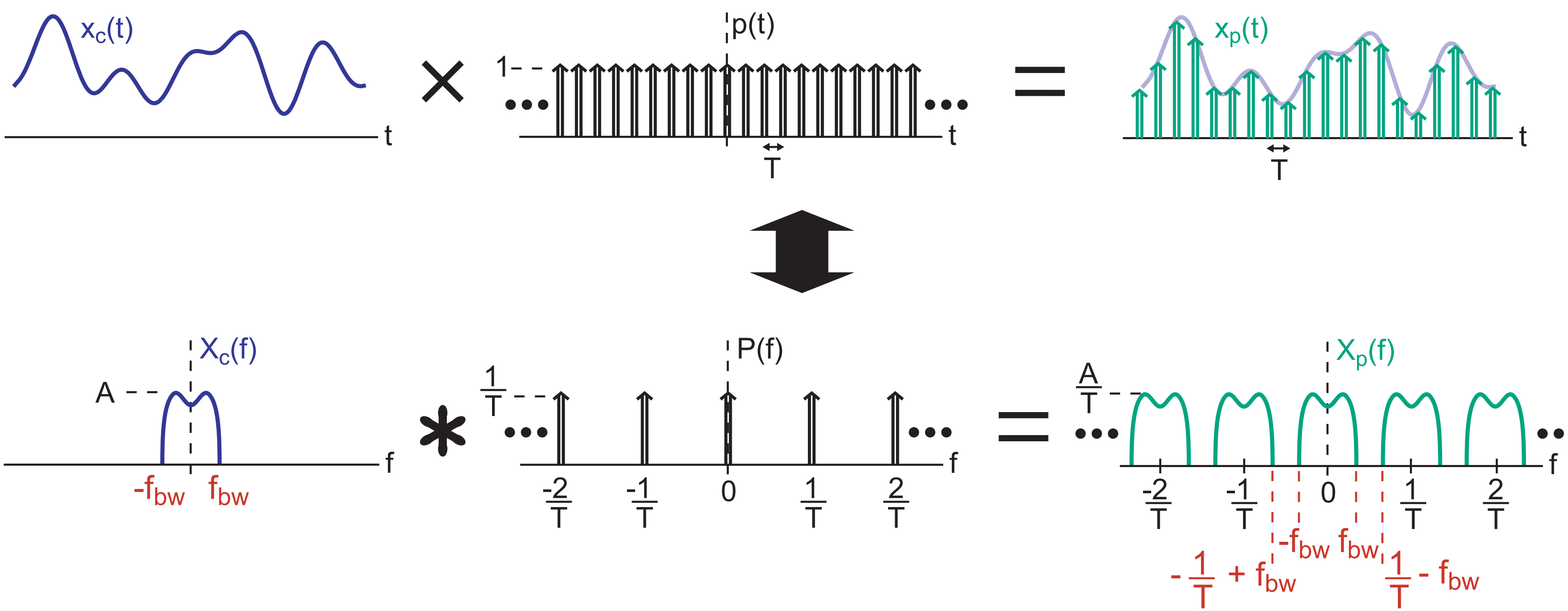
# Summary of Sampling Process




- Sampling leads to periodicity in frequency domain

**We need to avoid overlap of replicated signals in frequency domain (i.e., aliasing)**

# The Sampling Theorem

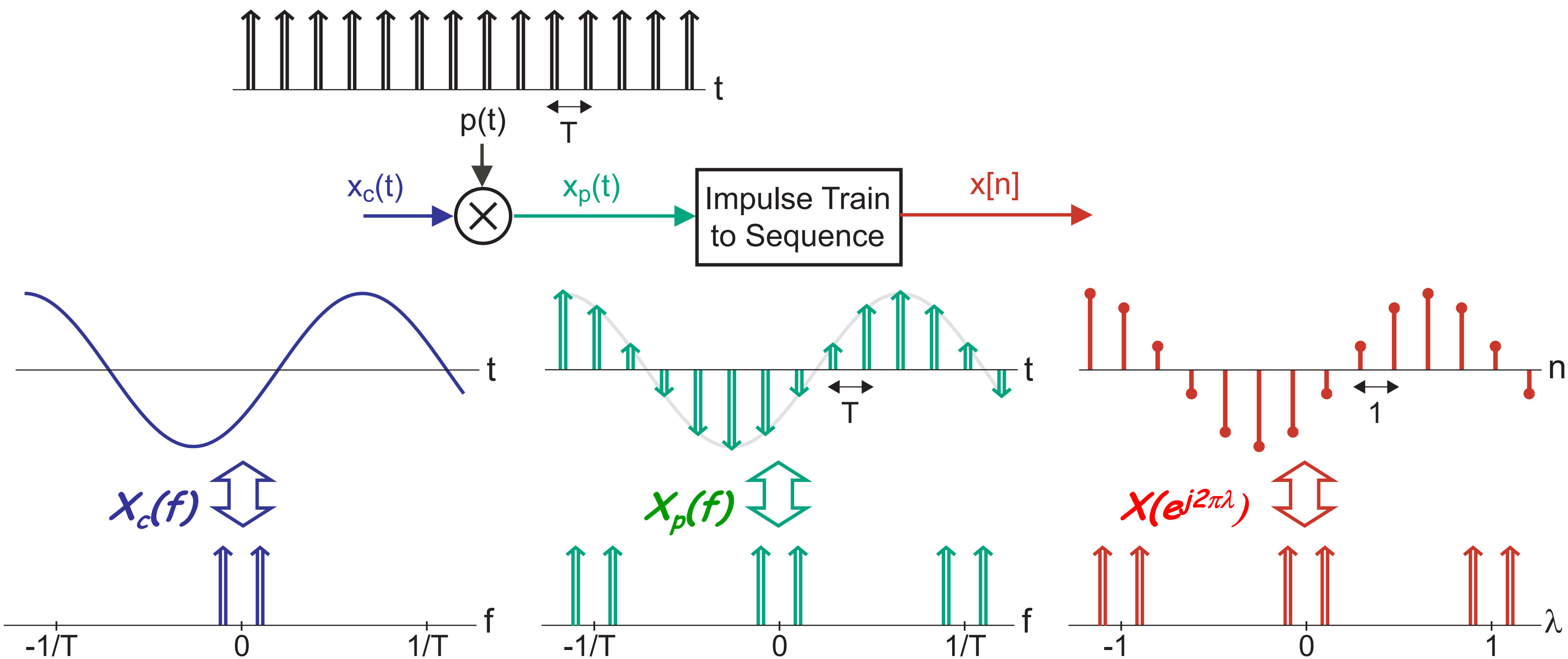



- **Overlap in frequency domain (i.e., aliasing) is avoided if:**

$$\frac{1}{T} - f_{bw} \geq f_{bw} \quad \Rightarrow \quad \boxed{\frac{1}{T} \geq 2 f_{bw}}$$

- **We refer to the minimum *1/T* that avoids aliasing as the *Nyquist* sampling frequency**

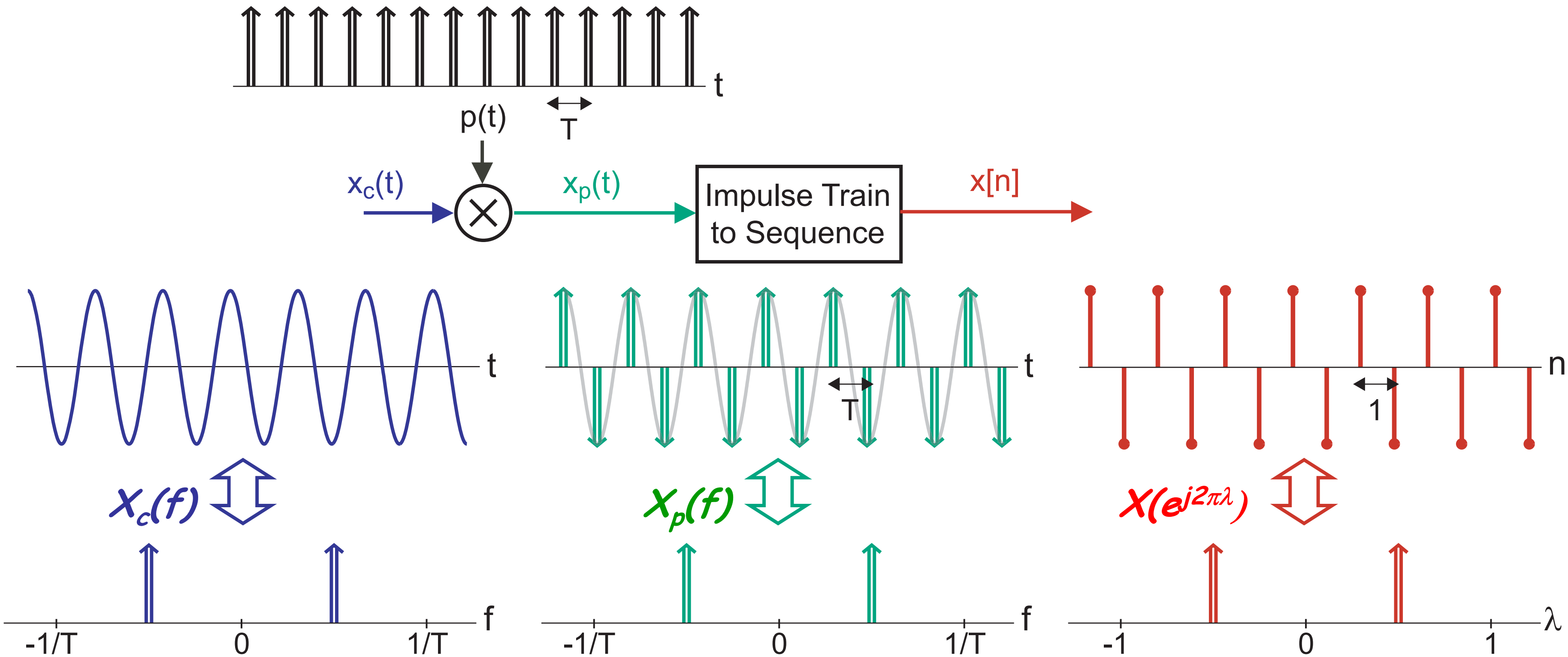
# Example: Sample a Sine Wave




**Sample rate is well *above* Nyquist rate**

- Time domain: resulting sequence maintains the same period as the input continuous-time signal
- Frequency domain: no aliasing

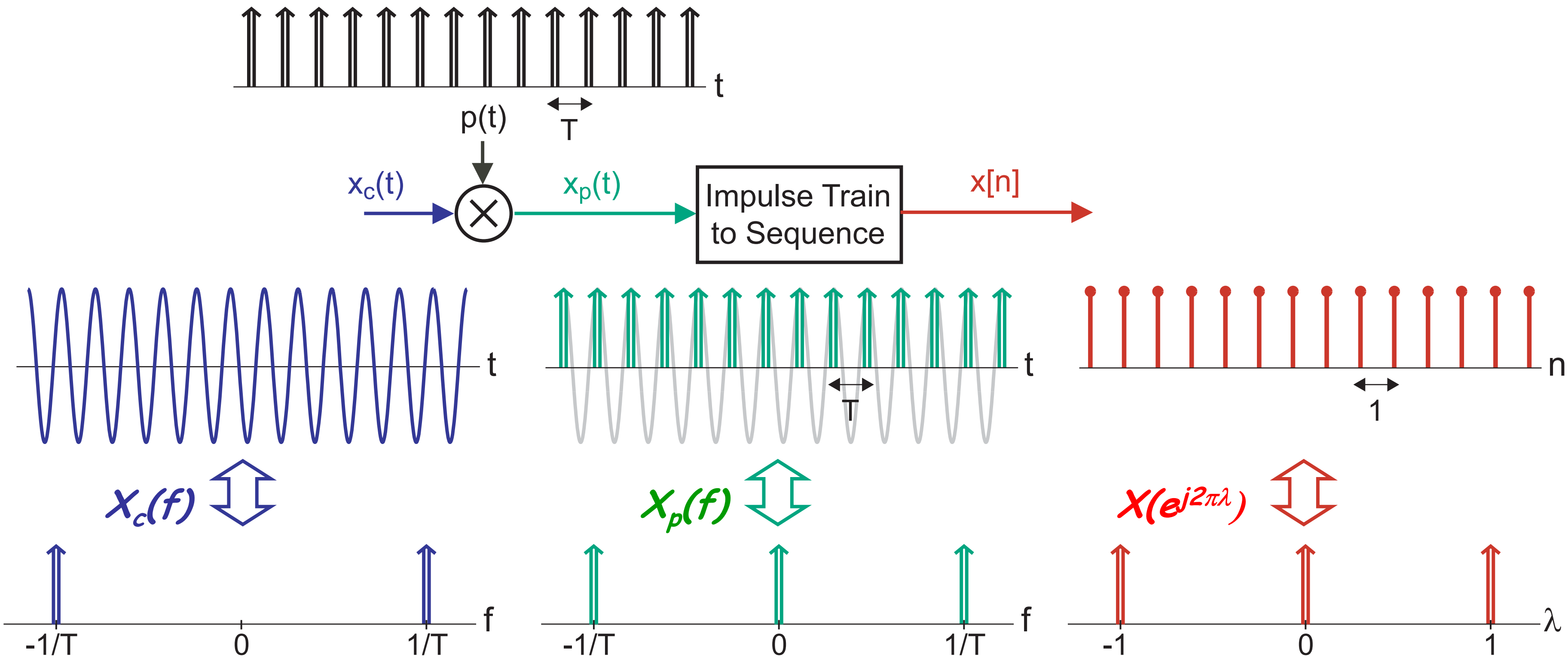
# Increase Input Frequency Further …




**Sample rate is *at* Nyquist rate**

- Time domain: resulting sequence still maintains the same period as the input continuous-time signal
- Frequency domain: no aliasing

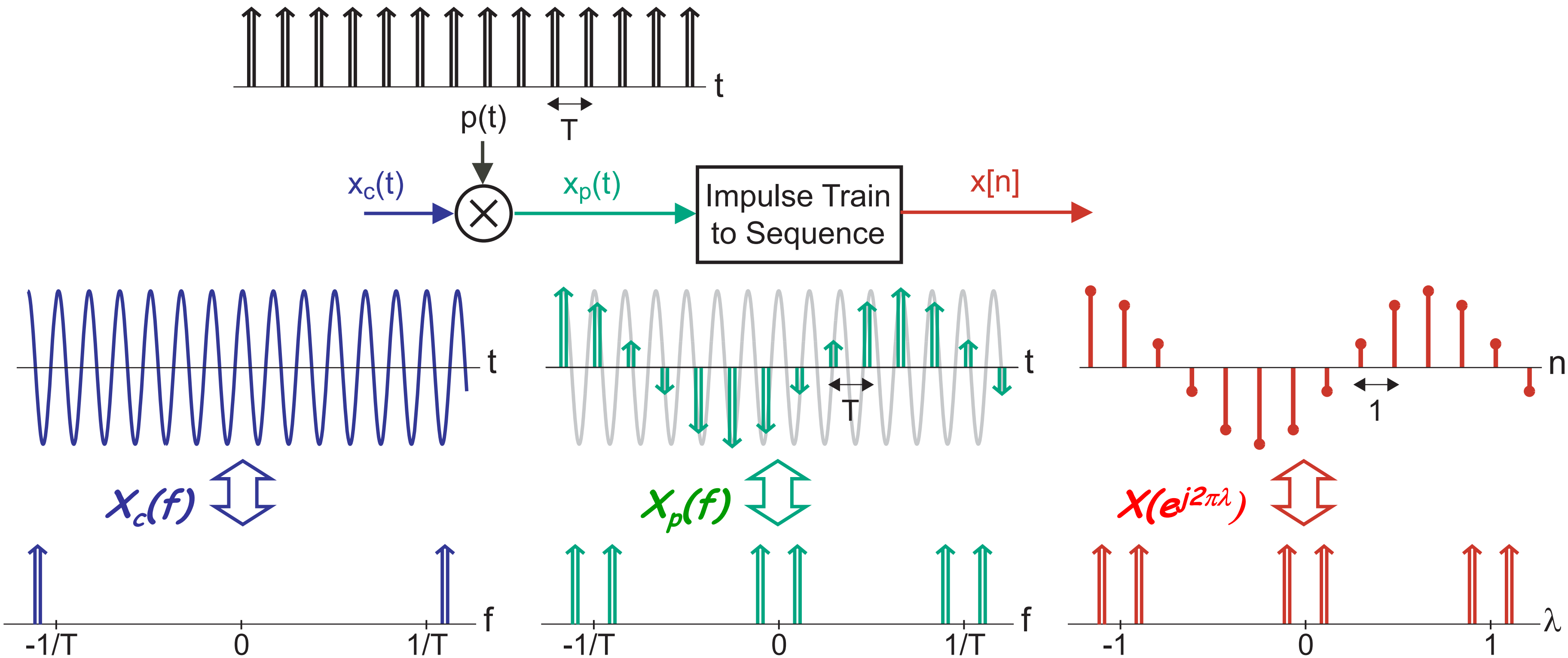
# Increase Input Frequency Further …




**Sample rate is at *half* the Nyquist rate**

- **Time domain: resulting sequence now appears as a DC signal!**
- **Frequency domain: aliasing to DC**

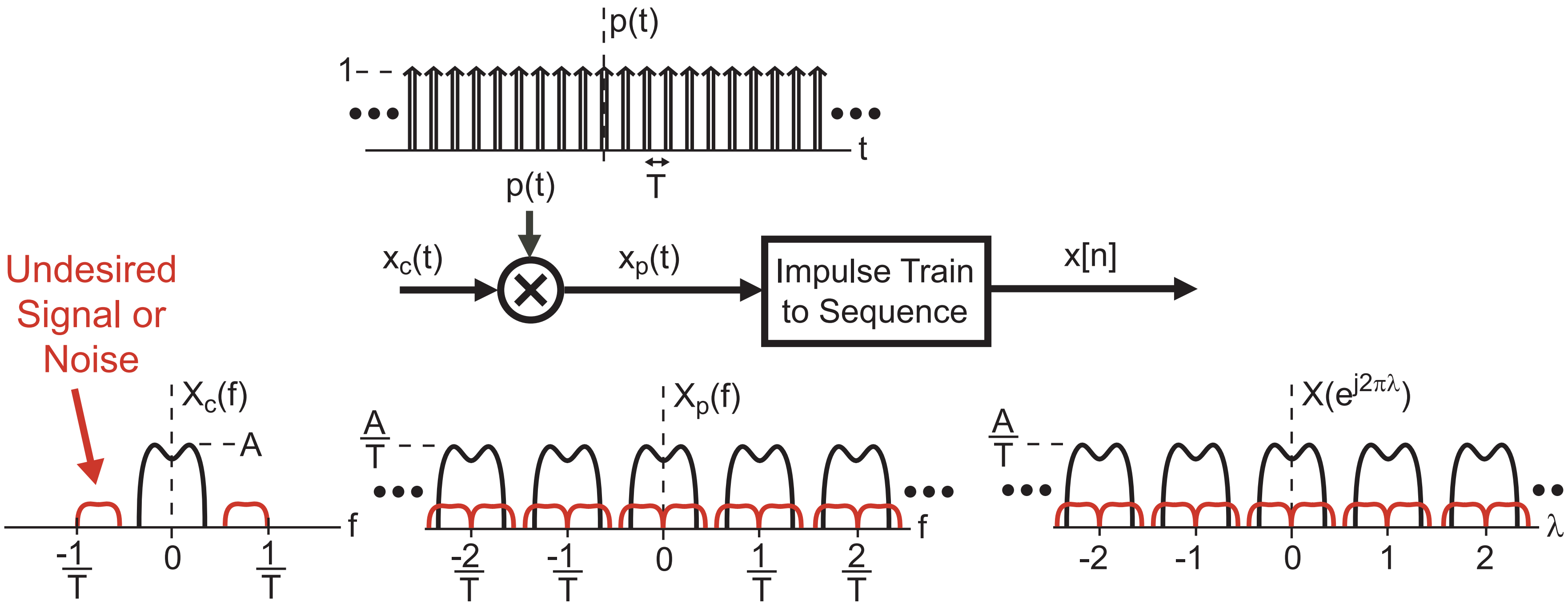
# Increase Input Frequency Further …




**Sample rate is well *below* the Nyquist rate**

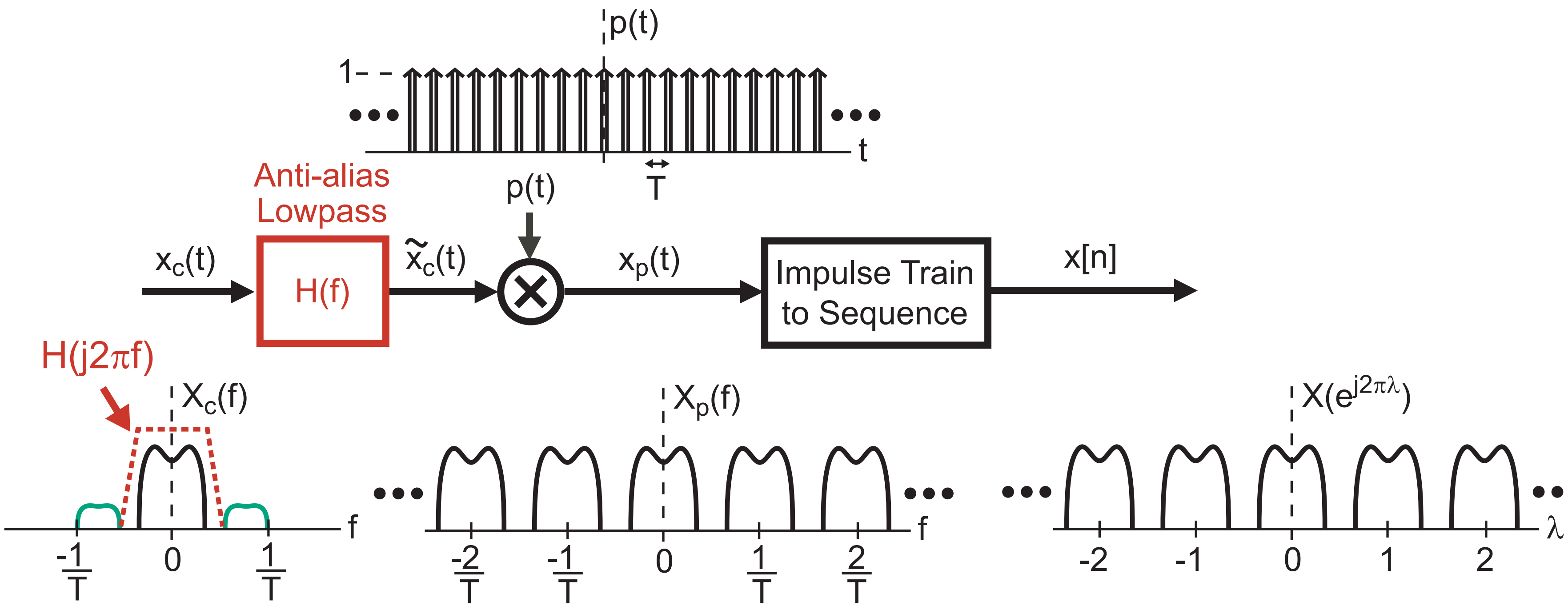
- **Time domain: resulting sequence is now a sine wave with a *different* period than the input**
- **Frequency domain: aliasing to lower frequency**

# The Issue of High Frequency Noise




- We typically set the sample rate to be large enough to accommodate full bandwidth of *signal*
- Real systems often introduce *noise* or other interfering signals at *higher* frequencies
  - Sampling causes this noise to *alias* into the desired signal band

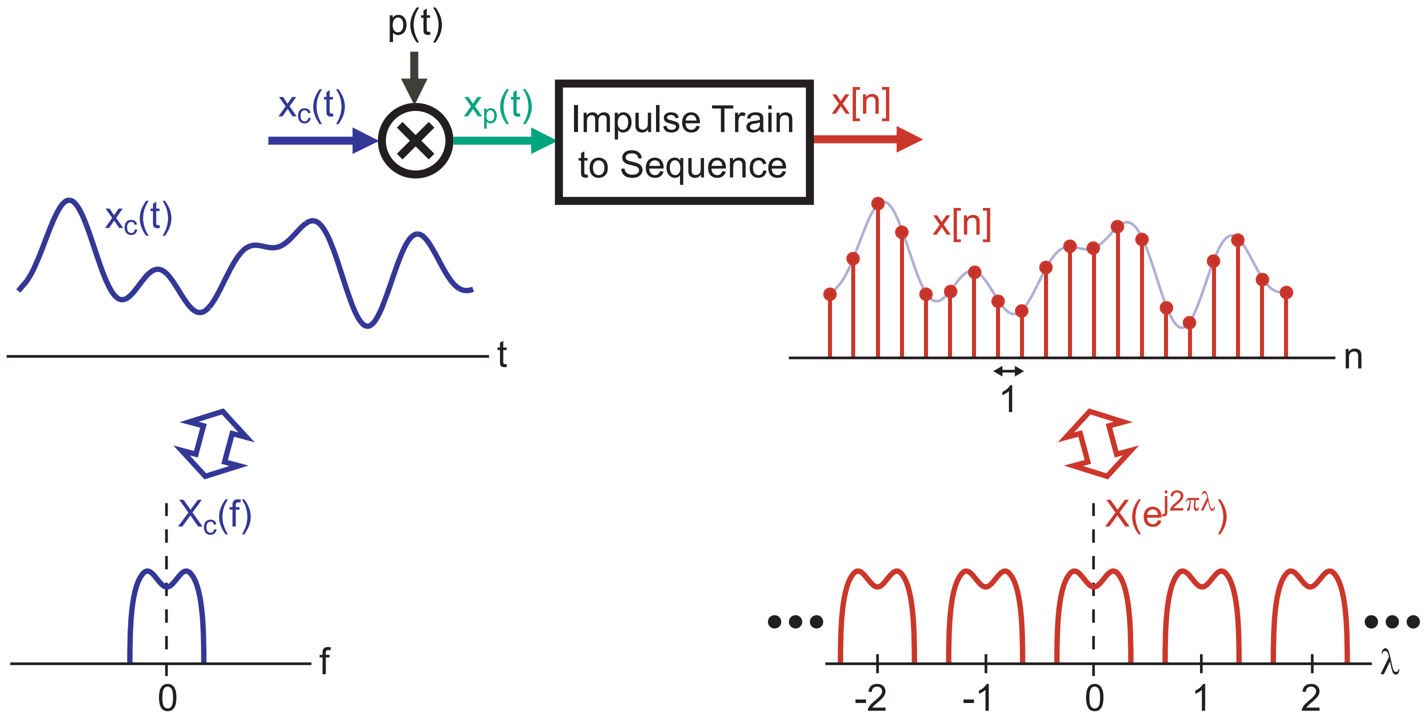
# Anti-Alias Filtering




- Practical A-to-D converters include a continuous-time filter *before* the sampling operation
  - Designed to filter out all noise and interfering signals above *1/(2T)* in frequency
  - Prevents aliasing

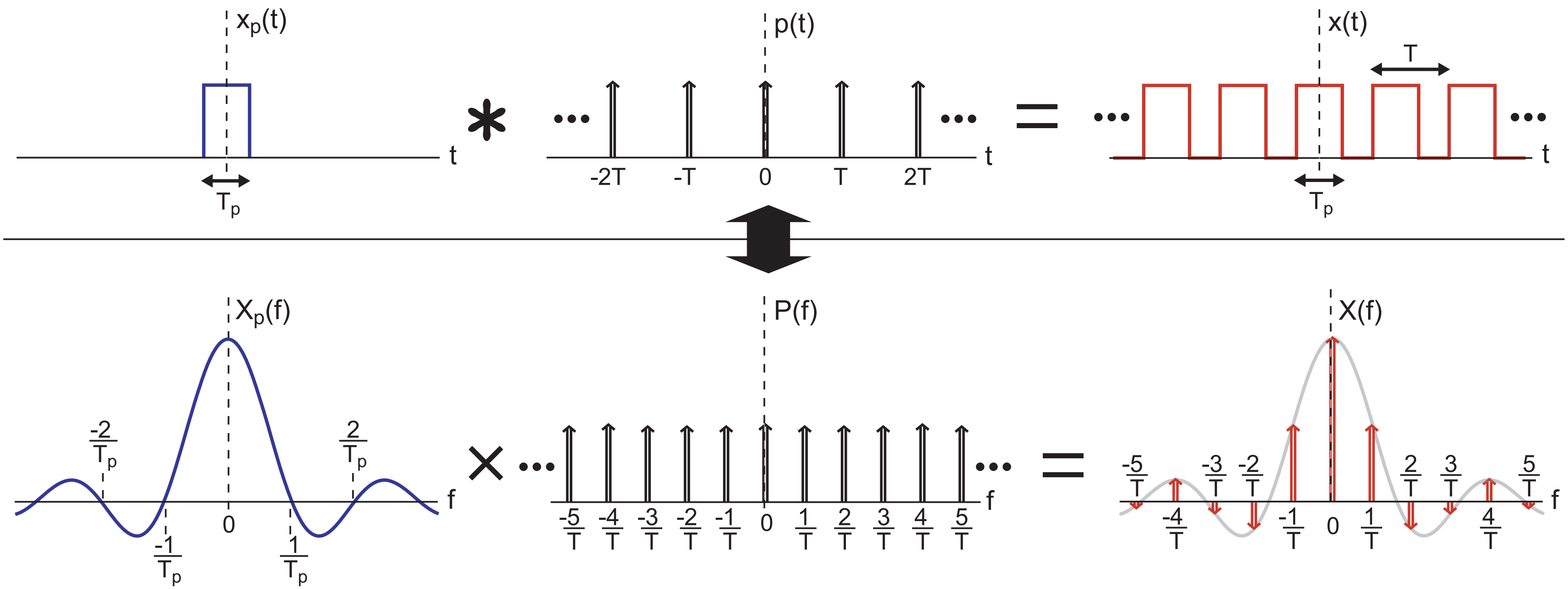
# Using the Impulse Train to Compare the FT, DTFT, and Fourier Series

# Relationship Between FT and DTFT




| | FT | DTFT |
|---|---|---|
| Time: | Continuous, Non-Periodic | Discrete, Non-Periodic |
| Freq: | Non-Periodic, Continuous | Periodic, Continuous |

# Relationship Between FT and Fourier Series




| | FT | Fourier Series |
|---|---|---|
| Time: | Continuous, Non-Periodic | Continuous, Periodic |
| Freq: | Non-Periodic, Continuous | Non-Periodic, Discrete |

# Summary

- The impulse train and its Fourier Transform form a very powerful analysis tool using *pictures*
  - Sampling, comparison of FT, DTFT, Fourier Series
- Sampling analysis:
  - Time domain: multiplication by an impulse train followed by re-scaling of time axis (and conversion to stem symbols)
  - Frequency domain: convolution by an impulse train followed by re-scaling of frequency axis
- Prevention of aliasing
  - Sample faster than Nyquist sample rate of signal bandwidth
  - Use anti-alias filter to cut out high frequency noise
- Up next: downsampling, upsampling, reconstruction