Two port analysis allows us to quickly calculate small signal gain from cascaded network stages

- So far, only purely resistive impedances have been considered
The Problem with Complex Impedances

- When complex impedances are considered (i.e., capacitors, inductors, and resistors), things get much more messy
  - Complex impedance calculations are time consuming
  - Capacitance between drain and gate of transistors complicates calculation effort further

Can we determine a faster analysis path to gain intuition?
**General Frequency Response for Amplifiers**

- **Midband gain can be calculated by assuming purely resistive impedances (as we have done so far)**
  - Large valued capacitors used for AC coupling will be shorts in this analysis
    - For DC coupled circuits, typically DC gain = Midband Gain
  - Small valued capacitors will be opens in this analysis

\[
\text{Note: } f (\text{Hz}) = \frac{\omega (\text{rad/s})}{2\pi}
\]
Our Focus Will Be on High Frequency Poles

- We are particularly interested in knowing the bandwidth of our amplifier circuit
  - Bandwidth is primarily set by the lowest frequency pole, \( w_0 \)
  - Additional attenuation occurs at frequencies beyond the amplifier bandwidth by higher frequency poles \( w_1, w_2, \) etc.

\[
\text{MidBand Gain } f (\text{Hz}) = \frac{w \text{ (rad/s)}}{2\pi}
\]
The Open Circuit Time Constant (OCT) technique allows us to quickly estimate the bandwidth of an amplifier circuit.

- We will see that it is most accurate when there is one dominant pole, $w_0$
  - This means that $w_1$, $w_2$, and higher poles are not close in frequency to $w_0$
    - This will hold for opamps and other circuits that operate in feedback
  - There is still considerable value to the OCT method in providing design intuition even when there is not just one dominant pole
# Short Circuit Time Constant Technique

- The Short Circuit Time Constant (SCT) technique allows us to quickly estimate the AC-coupled cutoff frequency, $w_{ac0}$.
  - This has many similarities to the OCT method, but we will not discuss in this class since
    - AC coupling is not used very often in integrated circuits due to the high cost of large valued capacitors.
    - When AC coupling is applied in integrated circuits, it is often quite easy to estimate the AC-coupled cutoff frequency since there are relatively few poles in the circuit related to AC-coupling.

Note:
\[ f (\text{Hz}) = \frac{w \text{ (rad/s)}}{2\pi} \]
Key Assumptions for the OCT Technique

Let us assume that the transfer function from $V_{in}$ to $V_{out}$ is

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{K}{(\tau_0 s + 1)(\tau_1 s + 1) \cdots (\tau_{n-1} s + 1)}$$

- Note that we are ignoring any AC-coupling poles/zeros
  - This implies that we are approximating DC gain = Midband gain
  - The OCT method does not require this assumption – it just simplifies the analysis to follow
- Note also that DC gain equals $K$ in the above transfer function
  - We see this by setting $s = 0$
Key Idea of the OCT Technique

- Assuming the transfer function from $V_{in}$ to $V_{out}$ is:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{K}{(\tau_0 s + 1)(\tau_1 s + 1)\cdots(\tau_{n-1} s + 1)}$$

- We can achieve a reasonable approximation of the bandwidth of the system by instead considering:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{K}{\left(\sum_{i=0}^{n-1} \tau_i\right) s + 1}$$

- Here $\tau_i$ are the “time constants” corresponding to the poles of the circuit network.
The OCT technique approximates the transfer function as:

\[
\frac{V_{out}(s)}{V_{in}(s)} = \frac{K}{(\sum_{i=0}^{n-1} \tau_i) s + 1}
\]

The estimated bandwidth is found by substituting \( s = jw_0 \) and solving for \( w_0 \) such that the magnitude is \( K/\sqrt{2} \)

\[
w_0 = \frac{1}{\sum_{i=0}^{n-1} \tau_i} \quad \Rightarrow \quad \left| \frac{V_{out}(w_0)}{V_{in}(w_0)} \right| = \left| \frac{K}{j1 + 1} \right| = \frac{K}{\sqrt{2}}
\]

Bandwidth estimate found by inverting the sum of time constants!
Why Is This Approximation Reasonable?

- Consider a second order example:

\[
\frac{V_{out}(w_0)}{V_{in}(w_0)} = \frac{K}{(j\tau_0 w_0 + 1)(j\tau_1 w_0 + 1)}
\]

- Expanding:

\[
\frac{V_{out}(w_0)}{V_{in}(w_0)} = \frac{K}{-\tau_0 \tau_1 w_0^2 + j(\tau_0 + \tau_1)w_0 + 1}
\]

- But notice (since the time constant values are > 0):

\[
j(\tau_0 + \tau_1)w_0 = j1 \Rightarrow \tau_0 w_0 < 1, \tau_1 w_0 < 1
\]

- In fact: \(\tau_0 \tau_1 w_0^2 \leq 0.25\)

- The worse case of \(\tau_0 \tau_1 w_0^2 = 0.25\) occurs when \(\tau_0 = \tau_1\):

\[
\left|\frac{V_{out}(w_0)}{V_{in}(w_0)}\right| = \frac{K}{j1 + 1 - 0.25} = \frac{K}{\sqrt{1.56}} \approx \frac{K}{\sqrt{2}}
\]

- The approximation will be better for \(\tau_0 \neq \tau_1\)
Key Issues For the OCT Approximation

- For the higher order transfer function

\[ \frac{V_{out}(s)}{V_{in}(s)} = \frac{K}{(\tau_0 s + 1)(\tau_1 s + 1) \cdots (\tau_{n-1} s + 1)} \]

- The OCT approximation for bandwidth is

\[ BW \approx \frac{1}{\sum_{i=0}^{n-1} \tau_i} \text{ rad/s} \]

- As hinted at by our second order example:
  - The OCT approximation will have much better accuracy if the time constants are different, and particularly if there is one dominant time constant
  - The bandwidth estimate by the OCT method is typically conservative (i.e., actual bandwidth > OCT estimate)
    - Complex poles can lead to actual bandwidth < OCT estimate

But how do we compute \( \sum_{i=0}^{n-1} \tau_i \) ?
OCT Method of Calculating the Sum of Time Constants

- OCT method calculates \( \sum_{i=0}^{n-1} \tau_i \) by the following steps:
  - Compute the effective resistance \( R_{thj} \) seen by each capacitor, \( C_j \), with other caps as open circuits
    - AC coupling caps are not included – considered as shorts
  - Form the “open circuit” time constant \( T_j = R_{thj}C_j \) for each capacitor \( C_j \)
  - Sum all of the “open circuit” time constants

- As proved by Richard Adler at MIT

\[
\sum_{i=0}^{n-1} \tau_i = \sum_{j=1}^{m} R_{thj}C_j
\]

- This implies that the sum of the transfer function pole time constants is the same as the sum of the open circuit time constants

\[
\Rightarrow BW \approx \frac{1}{\sum_{j=1}^{m} R_{thj}C_j} \text{ rad/s}
\]
How Do You Tell if a Cap is for AC coupling or OCT?

- In general, capacitors associated with AC coupling have the property that the amplifier gain *increases* as the capacitor goes from open to short
  - These capacitors are simply assumed to be shorts for the OCT analysis

- In general, capacitors used in the OCT calculation have the property that the amplifier gain *decreases* as the capacitor goes from open to short
  - These capacitors must all be considered in the OCT analysis
Example: Second Order RC Network

Transfer function of the above network:

\[
\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_1 C_2 + R_2 C_2)s + 1}
\]

The sum of the time constants from the poles of the above network are obtained by inspection of the first order coefficient in the above transfer function

\[
\Rightarrow BW \approx \frac{1}{\sum_{i=0}^{n-1} \tau_i} = \frac{1}{R_1 C_1 + R_1 C_2 + R_2 C_2} \text{rad/s}
\]

For more complex networks, the direct approach of explicitly calculating the transfer function is quite tedious
OCT Method Applied to Second Order RC Network

- Obtain the Thevenin resistance values seen by each capacitor with other capacitors as opens

\[ R_{th1} = R_1 \Rightarrow R_{th1}C_1 = R_1C_1 \]
\[ R_{th2} = R_1 + R_2 \Rightarrow R_{th2}C_2 = (R_1 + R_2)C_2 \]

- Bandwidth estimate from OCT method:

\[ \Rightarrow BW \approx \frac{1}{\sum_{j=1}^{m} R_{thj}C_j} = \frac{1}{R_1C_1 + (R_1 + R_2)C_2} \text{ rad/s} \]

- Note that OCT method agrees with estimate based on direct calculation of the transfer function, but is much faster!
Example: Common Source Amplifier

- Estimate the bandwidth of the above amplifier using the OCT method
  - What capacitances should be considered?
  - What Thevenin resistances must be calculated?
Key Capacitances for CMOS Devices

Top View

Side View

source to bulk cap: $C_{jsb} = \frac{C_j(0)}{\sqrt{1 + V_{SB}/|\Phi_B|}}$ WE + $\frac{C_{jsw}(0)}{\sqrt{1 + V_{SB}/|\Phi_B|}}$ (W + 2E)

drain to bulk cap: $C_{jsd} = \frac{C_j(0)}{\sqrt{1 + V_{DB}/|\Phi_B|}}$ WE + $\frac{C_{jsw}(0)}{\sqrt{1 + V_{DB}/|\Phi_B|}}$ (W + 2E)

overlap cap: $C_{ov} = WLD_{ox} + WC_{fringe}$

gate to channel cap: $C_{gc} = \frac{2}{3} C_{ox}W(L-2L_D)$

channel to bulk cap: $C_{cb}$ - ignore in this class
CMOS Hybrid-π Model with Caps (Device in Saturation)

\[ C_{gs} = C_{gc} + C_{ov} = \frac{2}{3} C_{ox} W (L - 2L_D) + C_{ov} \]

\[ C_{gd} = C_{ov} \]

\[ C_{sb} = C_{jsb} \quad \text{(area + perimeter junction capacitance)} \]

\[ C_{db} = C_{jdb} \quad \text{(area + perimeter junction capacitance)} \]
Of the above capacitors, only $C_{gs}$, $C_{gd}$, and $C_{db}$ must be considered

- $C_{sb}$ is grounded on both sides

Thevenin resistance calculations

- $C_{db}$: $R_{thd} \parallel R_d$
- $C_{gs}$ and $C_{gd}$: these involve new Thevenin resistance calculations
**OCT Thevenin Resistance Calculations**

- **C_{gs}**: Thevenin resistance between gate and source
  \[
  R_{th_{gs}} = \frac{R_S(1 + R_D/r_o) + R_G(1 + (g_{mb} + 1/r_o)R_S + R_D/r_o)}{1 + (g_m + g_{mb})R_S + (R_S + R_D)/r_o}
  \]

- **C_{gd}**: Thevenin resistance between gate and drain
  \[
  R_{th_{gd}} = (R_D + R_G)(1 - r_{ods}/r_o) + r_{ods}g_mR_G
  \]
  \[
  \text{where } r_{ods} = r_o|| \frac{R_D}{1 + (g_m + g_{mb})R_S}
  \]
Estimated bandwidth from OCT method:

\[
BW \approx \frac{1}{\sum_{j=1}^{m} R_{thj} C_j} = \frac{1}{(R_{thd} \parallel R_d) C_{db} + R_{thgd} C_{gd} + R_{thgs} C_{gs}} \text{ rad/s}
\]

The above calculations are straightforward given the Thevenin resistance formulas for \( R_{thd} \), \( R_{thgd} \), and \( R_{thgs} \).