Intro to Fourier Series

- Vector decomposition
- Even and Odd functions
- Fourier Series definition and examples
Review of Vector Decomposition

- Any vector can be decomposed into a set of appropriately weighted orthonormal basis vectors.
- Example:

\[ \hat{r} = a_x \hat{x} + a_y \hat{y} \]

\[ a_x = ??, \quad a_y = ?? \]
Calculation of Vector Weights

- Perform inner products with basis vectors
  \[ a_x = \hat{r} \cdot \hat{x}, \quad a_y = \hat{r} \cdot \hat{y} \]
- Example:
  \[ a_x = 1.3 \cdot 1 + .75 \cdot 0 = 1.3 \]
  \[ a_y = 1.3 \cdot 0 + .75 \cdot 1 = 0.75 \]
The Basis Vectors are Not Unique

\[ \hat{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \]

\[ \hat{y} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \]

\[ \hat{r} = [1.3 \ 0.75] \]

\[ \hat{r} = a_x \hat{x} + a_y \hat{y} \]

- **Inner product calculations:**

\[ a_x = 1.3 \cdot \frac{1}{\sqrt{2}} + .75 \cdot \frac{1}{\sqrt{2}} = \frac{2.05}{\sqrt{2}} \]

\[ a_y = 1.3 \cdot \frac{-1}{\sqrt{2}} + .75 \cdot \frac{1}{\sqrt{2}} = \frac{-0.55}{\sqrt{2}} \]
Observations on Basis Decomposition

- We can consider any vector as a sum of weighted orthonormal basis vectors.
- The weights are determined by an inner product calculations (also known as projections).
  - Consist of element-by-element multiplications followed by addition of the resulting products.

\[ \hat{r} = a_x \hat{x} + a_y \hat{y} \]

- Inner product calculations:
  \[ a_x = 1.3 \cdot \frac{1}{\sqrt{2}} + .75 \cdot \frac{1}{\sqrt{2}} = \frac{2.05}{\sqrt{2}} \]
  \[ a_y = 1.3 \cdot \frac{-1}{\sqrt{2}} + .75 \cdot \frac{1}{\sqrt{2}} = \frac{-0.55}{\sqrt{2}} \]
Can We Decompose Functions?

• Consider a periodic function such as a square wave

• Could we decompose the above waveform into a weighted sum of basis functions?

• If so, what would be a good choice for such basis functions?

• How would we calculate the weights?
Consider Sine Wave Basis Functions

- Suppose we consider sine waves of progressively increasing frequencies as our basis functions

\[
\begin{align*}
\sin(\omega_0 t) & \quad \cdots \quad t \\
\sin(2\omega_0 t) & \quad \cdots \quad t \\
\sin(3\omega_0 t) & \quad \cdots \quad t
\end{align*}
\]

- Check out the following Java applet demo:
  - Available at: http://www.falstad.com/fourier/
Issue: Sine Waves Are Limited

- A sine wave corresponds to an odd function

\[ \sin(\omega_0 t) \]

- Odd function definition:
  \[ f(t) = -f(-t) \]

- Adding odd functions together can only produce an odd function

\[
\begin{array}{c}
\cdots \\
\text{X} \\
\cdots \\
\end{array}
\]

\[
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\]
Consider Cosine Wave Basis Functions

- Even function definition:
  \[ f(t) = f(-t) \]

- Cosine waves are even functions
Combine Cosines and Sines

- If we use both cosine and sine waveforms as basis functions, we can realize both even and odd functions (and any combination)

**Even**

\[ \cos(\omega_0 t) \]
\[ \cos(2\omega_0 t) \]
\[ \cos(3\omega_0 t) \]

**Odd**

\[ \sin(\omega_0 t) \]
\[ \sin(2\omega_0 t) \]
\[ \sin(3\omega_0 t) \]
The Fourier Series

- A periodic waveform, \( x(t) \), with period \( T \) can be represented as an infinite sum of weighted cosine and sine waveforms

\[
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)
\]

where for \( n > 0 \):

\[
a_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \cos(n\omega_0 t) \, dt, \quad b_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \sin(n\omega_0 t) \, dt
\]

and where:

\[
\omega_0 = \frac{2\pi}{T}, \quad a_0 = \frac{1}{T} \int_{t_o}^{t_o+T} x(t) \, dt
\]
# Intuition for Fourier Series

- Compare Fourier Series to vector decomposition:

## Vector Decomp.

\[
\hat{r} = a_x \hat{x} + a_y \hat{y}
\]

\[
a_x = \hat{r} \cdot \hat{x},
\]

\[
a_y = \hat{r} \cdot \hat{y}
\]

## Fourier Series

\[
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_t) + b_n \sin(n\omega_t)
\]

\[
a_0 = \frac{1}{T} \int_{t_o}^{t_o+T} x(t) dt,
\]

\[
a_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \cos(n\omega_t) dt,
\]

\[
b_n = \frac{2}{T} \int_{t_o}^{t_o+T} x(t) \sin(n\omega_t) dt
\]

- The Fourier Series weight (i.e., \(a_n\) and \(b_n\)) calculations are analogous to vector inner products!
Sine Wave Example

\[ x(t) = K \sin(\omega_o t) \]

\[ a_0 = \frac{1}{T} \int_{t_o}^{t_o+T} K \sin(\omega_o t) \, dt = 0 \quad \text{(DC Average is 0)} \]

\[ a_n = \frac{2}{T} \int_{t_o}^{t_o+T} K \sin(\omega_o t) \cos(n\omega_o t) \, dt \]

\[ = \frac{K}{T} \int_{t_o}^{t_o+T} \sin((n-1)\omega_o t) + \sin((n+1)\omega_o t) \, dt = 0 \]

\[ b_n = \frac{2}{T} \int_{t_o}^{t_o+T} K \sin(\omega_o t) \sin(n\omega_o t) \, dt \]

\[ = \frac{K}{T} \int_{t_o}^{t_o+T} \cos((n-1)\omega_o t) + \cos((n+1)\omega_o t) \, dt = \begin{cases} K & (n = 1) \\ 0 & (n > 1) \end{cases} \]
Graphical View of Fourier Series (Sine)

\( x(t) = K \sin(\omega_0 t) \)

- We can plot Fourier coefficients as a function of index or frequency

Note: \( f = \frac{n}{T} \)
Fourier Series of Cosine

\[ x(t) = K \cos(\omega_0 t) \]

Note: \[ f = \frac{n}{T} \]
Fourier Series of Cosine with DC component

\[ x(t) = K \cos(\omega_0 t) + K \]

Note: \( f = \frac{n}{T} \)
Fourier Series of Phase-Shifted Cosine

\[ x(t) = K \cos(w_o t - \theta) \]

- Using a well known trigonometric identity:
  \[ K \cos(w_o t - \theta) = K \cos(\theta) \cos(w_o t) + K \sin(\theta) \sin(w_o t) \]
Vector View of Phase-Shifted Cosine

\[ x(t) = K \cos(w_\text{o}t - \theta) \]

- Using a well known trigonometric identity:
  \[ K \cos(w_\text{o}t - \theta) = K \cos(\theta) \cos(w_\text{o}t) + K \sin(\theta) \sin(w_\text{o}t) \]

\[ Q = \sin(w_\text{o}t) \]

\[ I = \cos(w_\text{o}t) \]

K \sin(\theta) \quad K \cos(w_\text{o}t - \theta)
**Square Wave Example**

- **By inspection:**
  - DC average = 0 , \( a_0 = 0 \)
  - \( x(t) \) is odd , \( a_n = 0 \) (\( n \neq 1 \))

\[
b_n = \frac{2}{T} \left( \int_{-T/2}^{0} -A \sin(n\omega_o t) \, dt + \int_{0}^{T/2} A \sin(n\omega_o t) \, dt \right)
\]

\[
= \frac{4A}{T} \int_{0}^{T/2} \sin(n\omega_o t) \, dt = \frac{4A}{T} \frac{1}{n\omega_o} \left( - \cos(n\omega_o T/2) + 1 \right)
\]

\[
= \frac{4A}{T} \frac{T}{n2\pi} \left( - \cos(n \frac{2\pi}{T} T/2) + 1 \right) = \frac{2A}{n\pi} \left( - \cos(n\pi) + 1 \right)
\]

\[\Rightarrow b_n = \frac{4A}{n\pi} \text{ for } n \text{ odd, } \quad b_n = 0 \text{ for } n \text{ even}\]
Summary

• Vector decomposition provides a nice starting point for understanding Fourier Series
  - Vector decomposition into a sum of weighted basis vectors

• Fourier Series decomposes periodic waveforms into an infinite sum of weighted cosine and sine functions
  - We can look at waveforms either in 'time' or 'frequency'
  - Useful tool: even and odd functions

• Some issues we will deal with next time
  - Fourier Series definition covered today is not very compact
    • We will look at a simpler formulation based on complex exponentials
  - Fourier Series only deals with periodic waveforms
    • We will introduce the Fourier Transform to deal with non-periodic waveforms